

# The Einstein-Maxwell system, Ward identities, and the Vilkovisky construction

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## Abstract

The gauge fixing dependence of the one-loop effective action of quantum gravity in the proper-time representation is investigated for a space of arbitrary curvature, and the investigation is extended to Maxwell-Einstein theory. The construction of Vilkovisky and DeWitt for removal of this dependence is then considered in general gauges, and it is shown that nontrivial criteria arising from a Ward identity of the theory must be obeyed by the regularization scheme, if the construction is to remove the gauge dependence of quadratic and quartic divergences. The results apply also to non-Abelian gauge theories; they are used to address the question of gauge dependence of asymptotic freedom arising through internal graviton lines at one-loop order as suggested by Robinson and Wilczek.

*Keywords:* Quantum fields in curved space.

*PACS numbers* 04.62.+v, 11.10.Gh, 11.15.-q.

# 1 Introduction

In an influential paper, Robinson and Wilczek [1] suggested the possibility of asymptotic freedom arising in a gauge theory, considered an effective field theory in the sense of Weinberg [2] and Donoghue [3], through the quadratic divergences of one-loop Feynman integrals involving internal graviton lines. It was subsequently pointed out by Pietrykowski [4] that the effect is gauge dependent and that it vanishes in a class of gauges. This was later confirmed in [5]. The topic has recently generated much active interest and controversy in the literature [6].

The results in [1]-[6] were based on flat-space calculations, which means that the background gravitational field is not a solution of the Einstein equation, and this in its turn leads to a gauge-dependent result, since off-shell quantities in a quantum field theory depend on the gauge. A gauge-independent construction of the off-shell effective action was developed by Vilkovisky [7] and extended by DeWitt [8] (we shall for brevity refer to this method below as the Vilkovisky construction). Their method was recently applied in connection with the problem treated in [1]-[6] by Toms [9] using the Schwinger-DeWitt proper-time representation of the effective action [10], [11] and by He, Wang and Xianyu [12] and Tang and Wu [13], using momentum space integration. They all obtain different numerical results.

The scope of the present investigation is threefold:

- The problem of [1] is considered in a space with arbitrary curvature by means of the Schwinger-DeWitt proper-time representation of the effective action of the gravitational field in arbitrary gauges; thus the Einstein equation for the background metric may be applied, formally eliminating the gauge dependence of the effective action, and also the consequences in general of having a background metric that is not a solution of the Einstein equation can be found.
- The Vilkovisky construction is investigated in detail on the one-loop level in general gauges rather than the so-called Landau-DeWitt gauge [14] to which most previous investigations have been restricted. It is found that gauge independence of the one-loop effective action is a consequence of a certain Ward identity, and the Vilkovisky construction can thus only be applied in connection with regularization schemes where this Ward identity is not violated.
- This method is then applied to the Maxwell-Einstein system, and it is found that its one-loop effective action is made gauge-invariant off-shell at second order in the gravitational coupling  $\kappa$  by the Vilkovisky construction of pure quantum gravity, whereas the version of the Landau-DeWitt gauge used in [9], [12] and [13] is relevant for the off-shell effective action at fourth order in  $\kappa$ .

Because the topic has generated so much controversy a rather detailed exposition has been used. Only coupled Maxwell-Einstein fields are considered,

but the conclusions carry over almost verbatim to non-Abelian gauge fields coupled to gravity.

The layout of this article is the following: In sec. 2 we consider a general gauge theory and carry out the Vilkovisky construction in the one-loop approximation, showing how a general Ward identity is formally valid and implies through a partial cancellation of the gauge field and ghost contribution that the effective action is independent of the gauge condition. In sec. 3 we consider the one-loop effective action of pure quantum gravity in an arbitrary background metric and in a class of gauges more general than the Feynman gauge, showing that the Schwinger-DeWitt proper time representation can be used also in this case, and that the gravitational heat kernel obeys a Ward identity that determines the gauge dependence of the effective action in the case where the background metric is not a solution of the Einstein equation. The Vilkovisky construction is also carried out in this case and shown not to eliminate the gauge dependence of all quadratically divergent terms. In sec. 4 the Maxwell-Einstein system is considered, and it is found that the considerations on the gauge dependence of the effective action obtained in sec. 3 carry over to this case also, with the replacement  $\mathcal{G}^{\mu\nu} \rightarrow \mathcal{G}^{\mu\nu} - \mathcal{T}^{\mu\nu}$ , where  $\mathcal{G}^{\mu\nu}$  is the Einstein tensor and  $\mathcal{T}^{\mu\nu}$  the energy-momentum tensor of the background gauge field. Also a class of generalized gauges is introduced involving the background gauge field and field strength, following [9], [12] and [13], and it is proven that the contribution at second order in the gravitational coupling  $\kappa$  formally vanishes by the Ward identities of the theory when the ghost contributions are taken into account, but that this is upset by quadratic divergences with proper time regularization. We also apply the Vilkovisky construction to this case, verifying in detail how it removes the gauge parameter dependence of the induced action. Finally sec. 5 contains evaluation of the effective action either by momentum-space integration or by the proper-time method. Appendix A gives technical details on proper-time regularization while Appendix B contains the outline of the Vilkovisky construction of Maxwell-Einstein theory in next-lowest order in  $\kappa$ .

The following conventions have been used: The metric in Minkowski space is  $\eta_{\mu\nu} = (-+++)$ , and the sign of the Riemann tensor is chosen such that the Hilbert action is:

$$S_H = \frac{1}{\kappa^2} \int d^4x \sqrt{-g} R \quad (1)$$

where  $\kappa = \sqrt{8\pi G}$  is the gravitational coupling constant, with  $G$  denoting Newton's constant.

## 2 General gauge theory

The effective action  $\Gamma[\phi]$  of a field theory with classical action  $S[\varphi]$  is given by the path integral through:

$$e^{i\Gamma[\phi]} = \int [\mathcal{D}\varphi] \exp(i\Gamma[\phi]_k(\phi^k - \varphi^k) + S[\varphi]). \quad (2)$$

We use the condensed notation of DeWitt [11], where the label of the background field  $\phi$  and the integration variable  $\varphi$  indicates both space-time variable, tensor indices and group indices. From (2) follows in the one-loop approximation:

$$\Gamma[\phi] \simeq S[\phi] - \frac{i}{2} \text{Tr} \log \Delta[\phi]. \quad (3)$$

The propagator  $\Delta[\phi]^{ik}$  is defined by:

$$\Delta[\phi]^{ik} S[\phi]_{,kl} = \delta^i_l. \quad (4)$$

Vilkovisky [7] and De Witt [8] use in (2) instead of the difference between the background field  $\phi$  and the integration variable  $\varphi$  the geodesic interval in field space, given a suitable metric tensor in field space, and this leads in (2) to the replacement:

$$\phi^k - \varphi^k \rightarrow \phi^k - \varphi^k - \frac{1}{2} \Gamma^k_{lm}[\phi] (\phi^l - \varphi^l) (\phi^m - \varphi^m) + \dots \quad (5)$$

with  $\Gamma^k_{lm}$  components of a connection in field space. Then (3) becomes:

$$\Gamma[\phi] \simeq S[\phi] - \frac{i}{2} \text{Tr} \log \Delta[\phi] - \frac{i}{2} S[\phi]_{,m} \Gamma^m_{kl}[\phi] \Delta[\phi]^{lk} + \dots \quad (6)$$

Gauge transformations are:

$$\delta\phi^i = R^i_{\alpha}[\phi] \delta\lambda^{\alpha} \quad (7)$$

with  $\delta\lambda^{\alpha}$  an infinitesimal parameter, and where the gauge invariance of the classical action is expressed by:

$$S[\phi]_{,i} R^i_{\alpha}[\phi] = 0. \quad (8)$$

From (8) follows:

$$R^i_{\alpha}[\phi] S[\phi]_{,ij} + S[\phi]_{,i} R^i_{\alpha}[\phi]_{,j} = 0 \quad (9)$$

so  $S[\phi]_{,ij}$  is degenerate and not invertible on the mass shell, where  $S[\phi]_{,i} = 0$ . The gauge transformation generators  $R^i_{\alpha}[\phi]$  fulfill the structure relations:

$$R^i_{\alpha,j} R^j_{\beta} - R^i_{\beta,j} R^j_{\alpha} = c^{\gamma}_{\alpha\beta} R^i_{\gamma} \quad (10)$$

where  $c^{\gamma}_{\alpha\beta}$  are generalized structure constants .

Gauge conditions are  $\chi^\alpha[\phi]$  that are taken linear in  $\phi$ , and the degeneracy of  $S[\phi]_{,ij}$  is lifted by substituting:

$$S[\phi]_{,ij} \rightarrow S[\phi]_{,ij} + \chi^\alpha_{,i} c_{\alpha\beta} \chi^\beta_{,j} \quad (11)$$

with  $c_{\alpha\beta}$  a constant, symmetric matrix, and (6) is in a gauge theory replaced by:

$$\Gamma[\phi] \simeq S[\phi] - \frac{i}{2} \text{Tr} \log \Delta[\phi] + i \text{Tr} \log Q^{-1} - \frac{i}{2} S[\phi]_{,i} \tilde{\Gamma}^i_{kl}[\phi] \Delta[\phi]^{lk} \quad (12)$$

with terms containing more than one power of  $S[\phi]_{,i}$  disregarded and with:

$$Q^\alpha_\beta = \chi^\alpha_{,i} R^i_\beta, \quad \det Q^\alpha_\beta \neq 0 \quad (13)$$

where the new connection coefficients  $\tilde{\Gamma}^i_{kl}$  were constructed by Vilkovisky [7]. After gauge breaking (9) is replaced by:

$$R^i_\alpha[\phi] S[\phi]_{,ij} + S[\phi]_{,i} R^i_\alpha[\phi]_{,j} = Q^\beta_\alpha c_{\beta\gamma} \chi^\gamma_{,j}. \quad (14)$$

Multiplying this relation by  $(Q^{-1})^\alpha_\gamma \Delta[\phi]^{jk}$  one obtains the Ward identity [15]:

$$c_{\alpha\beta} \chi^\beta_{,j} \Delta[\phi]^{jk} = (Q^{-1})^\beta_\alpha (R^k_\beta[\phi] + S[\phi]_{,i} R^i_{\beta,j} \Delta^{jk}[\phi]). \quad (15)$$

We introduce the operator:

$$N_{\alpha\beta} = R^i_\alpha \gamma_{ij} R^j_\beta \quad (16)$$

with the inverse  $N^{\alpha\beta}$ . Here  $\gamma_{mn}$  is the metric in field space, with the inverse metric  $\gamma^{mn}$ . We also define the projection operators:

$$\Pi_{mn} = \gamma_{mn} - \gamma_{mi} R^i_\alpha N^{\alpha\beta} R^k_\beta \gamma_{kn} \quad (17)$$

with:

$$R^m_\alpha \Pi_{mn} = 0. \quad (18)$$

The connection in field space after gauge fixing  $\tilde{\Gamma}^r_{mn}$  is according to Vilkovisky [7] given by:

$$\begin{aligned} \tilde{\Gamma}^r_{mn} &\simeq \Gamma^r_{mn} - \gamma_{mk} R^k_\alpha N^{\alpha\beta} \mathcal{D}_n R^r_\beta - \gamma_{nk} R^k_\alpha N^{\alpha\beta} \mathcal{D}_m R^r_\beta \\ &+ \frac{1}{2} \gamma_{mi} R^i_\alpha N^{\alpha\gamma} (R^j_\delta \mathcal{D}_j R^r_\gamma + R^j_\gamma \mathcal{D}_j R^r_\delta) N^{\delta\beta} R^k_\beta \gamma_{kn} \end{aligned} \quad (19)$$

with the covariant derivative defined through:

$$\mathcal{D}_n R^i_\alpha = R^i_{\alpha,n} + \Gamma^i_{mn} R^m_\alpha. \quad (20)$$

and with  $\Gamma^r_{mn}$  the Christoffel connection components in field space before gauge fixing. The following form of (19) turns out to be convenient:

$$\begin{aligned} \tilde{\Gamma}^r_{mn} &\simeq \Gamma^r_{ij} \Pi^i_m \Pi^j_n - \frac{1}{2} R^r_{\alpha,n} N^{\alpha\beta} R^k_\beta \gamma_{km} - \frac{1}{2} R^r_{\alpha,m} N^{\alpha\beta} R^k_\beta \gamma_{kn} \\ &- \frac{1}{2} \Pi^s_m R^r_{\alpha,s} N^{\alpha\beta} R^k_\beta \gamma_{kn} - \frac{1}{2} \Pi^s_n R^r_{\alpha,s} N^{\alpha\beta} R^k_\beta \gamma_{km}. \end{aligned} \quad (21)$$

The last term of (12), using the modified connection (21), becomes:

$$-\frac{i}{2}S[\phi]_{,j}\Delta[\phi]^{nm}\left(\Gamma^j_{qs}\Pi^q_m\Pi^s_n - R^j_{\alpha,n}N^{\alpha\beta}R^k_{\beta}\gamma_{km} - \Pi^s_n R^j_{\alpha,s}N^{\alpha\beta}R^k_{\beta}\gamma_{km}\right). \quad (22)$$

Picking a gauge where the propagator is restricted by the equation:

$$R^k_{\alpha}\gamma_{kn}\Delta[\phi]^{nm} = 0 \quad (23)$$

(the Landau-De Witt gauge) this expression reduces to:

$$-\frac{i}{2}S[\phi]_{,j}\Delta[\phi]^{nm}\Gamma^j_{mn}. \quad (24)$$

It is next verified that the gauge dependence of (12) has been eliminated by addition of the last term. Gauge dependence occurs through the gauge fixing function  $\chi^\alpha$  and also through the matrix elements  $c_{\alpha\beta}$ , but it is sufficient to consider variation of  $\chi^\alpha$ , since the arbitrariness connected to  $c_{\alpha\beta}$  can be absorbed in the gauge fixing function.

From (12) one finds:

$$\frac{\delta}{\delta\chi^\alpha_{,j}}\left(-\frac{i}{2}\text{Tr log } \Delta[\phi]\right) = ic_{\alpha\beta}\chi^\beta_{,k}\Delta^{kj}[\phi]. \quad (25)$$

Assuming here and henceforth that the Ward identity (15) can be applied one gets from (25):

$$\frac{\delta}{\delta\chi^\alpha_{,j}}\left(-\frac{i}{2}\text{Tr log } \Delta[\phi]\right) = i(Q^{-1})^\delta_\alpha(R^i_\delta + S[\phi]_{,k}R^k_{\delta,l}\Delta[\phi]^{lj}). \quad (26)$$

where in (26) the first term on the right-hand is cancelled by the ghost term derivative:

$$\frac{\delta}{\delta\chi^\alpha_{,j}}(i\text{Tr log } Q^{-1}) = -i(Q^{-1})^\delta_\alpha R^j_\delta. \quad (27)$$

(this could be upset by the regularization scheme). Also we find:

$$\frac{\delta}{\delta\chi^\alpha_{,j}}\Delta[\phi]^{nm} \simeq -\Delta[\phi]^{mj}(Q^{-1})^\gamma_\alpha R^n_\gamma - \Delta[\phi_0]^{nj}(Q^{-1})^\gamma_\alpha R^m_\gamma \quad (28)$$

by the Ward identity (15) and where terms involving  $S[\phi]_{,k}$  were disregarded, and thus we get from (22) after some manipulations and using (16) and (18) and also the structure relations (10) and the gauge invariance of the classical action:

$$\frac{\delta}{\delta\chi^\alpha_{,j}}\left(-\frac{i}{2}S[\phi]_{,m}\tilde{\Gamma}^m_{kl}[\phi_0]\Delta[\phi]^{lk}\right) \simeq -iS[\phi]_{,k}R^k_{\beta,n}\Delta[\phi_0]^{nj}(Q^{-1})^\beta_\alpha. \quad (29)$$

The sum of (26), (27) and (29) vanishes (the partial cancellation between (27) and (29) could again be upset by the regularization scheme).

In summary, it was verified that the Vilkovisky construction as expected formally removes the gauge parameter dependence of the effective action at

one-loop order. In the course of this proof, conditions for the regularization scheme to be used were obtained: It should not upset the cancellation between (26), (27) and (29). It will be found in the following sections for the cases of quantum gravity and the Einstein-Maxwell system that these requirements are nontrivial and indeed are violated by quartic and quadratic divergences in the Schwinger-DeWitt proper time representation of the effective action with a lower cut-off in the proper time variable.

### 3 Pure quantum gravity in the one-loop approximation

In this section we investigate the gauge parameter dependence of the one-loop effective action of pure quantum gravity with an arbitrary background metric, using the Schwinger-DeWitt proper time representation. The one-loop effective action in a general field theory is determined from (3), (6) or (12), where in the proper time representation:

$$-\frac{i}{2}\text{Tr} \log \Delta[\phi] = -\frac{i}{2}\text{Tr} \int_0^\infty \frac{d\tau}{\tau} e^{i\tau\Delta^{-1}[\phi]} \quad (30)$$

where  $\tau$  is the proper time, with a corresponding expression for the ghost contribution  $i\text{Tr} \log Q^{-1}$ . This method is convenient because of the Campbell-Baker-Hausdorff identity:

$$\delta e^A = \int_0^1 dt e^{tA} \delta A e^{(1-t)A} \quad (31)$$

with  $A$  an arbitrary operator and  $\delta A$  an infinitesimal variation that does not commute with  $A$ ; this identity is convenient for a perturbative expansion of the effective action. (30) is an exact but possibly divergent representation of the effective action, where a regularization is achieved by a modification of the proper time integral at the lower end.

The metric tensor  $g_{\mu\nu}$  is split according to:

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + \kappa h_{\mu\nu} \quad (32)$$

with  $g_{\mu\nu}$  a classical background metric field, while  $h_{\mu\nu}$  is the quantum fluctuation field. A coordinate transformation implies for  $h_{\mu\nu}$ :

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \xi_{\mu;\nu} + \xi_{\nu;\mu} + \kappa(h_{\mu\lambda}\xi^\lambda_{;\nu} + h_{\nu\lambda}\xi^\lambda_{;\mu} + \xi^\lambda h_{\mu\nu;\lambda}) + O(\kappa^2). \quad (33)$$

Here and elsewhere, covariant derivative is indicated by a semicolon, and if the covariant derivative is with respect to a variable  $x'$ , then the index following the semicolon carries a prime, etc.. The Hilbert action (1) has the linear term in  $h_{\mu\nu}$ :

$$S_H^{(1)} = -\frac{1}{\kappa} \int d^4x \sqrt{-g} h_{\mu\nu} \mathcal{G}^{\mu\nu} \quad (34)$$

with  $\mathcal{G}^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$  the Einstein tensor of the background metric, where  $R^{\mu\nu}$  the Ricci tensor and  $R = g_{\mu\nu}R^{\mu\nu}$  the curvature scalar, and also the following quadratic term in  $h_{\mu\nu}$ :

$$S_H^{(2)} = \frac{1}{2} \int d^4x \sqrt{-g} \left( -\frac{1}{2} h^\lambda{}_\lambda \mathcal{G}^{\mu\nu} h_{\mu\nu} + h^\mu{}_\rho \mathcal{G}^{\rho\nu} h_{\mu\nu} + R^{\mu\lambda} h^\nu{}_\lambda h_{\mu\nu} - \frac{1}{2} h^\mu{}_\mu R^{\lambda\rho} h_{\lambda\rho} \right. \\ \left. + \frac{1}{2} h_{\mu\nu} g^{\mu\nu} (h^{\lambda\rho}{}_{;\lambda;\rho} - h^\lambda{}_{\lambda;\rho}{}^{;\rho}) - \frac{1}{2} h^{\mu\nu} (h_{\mu\lambda;\nu}{}^{;\lambda} + h_{\nu\lambda;\mu}{}^{;\lambda} - h_{\mu\nu}{}^{;\lambda}{}_{;\lambda} - h^\lambda{}_{\lambda;\mu;\nu}) \right). \quad (35)$$

In order to quantize the gravitational field one adds to (35) a gauge breaking term:

$$S_{GB} = -\frac{1}{2} \frac{1}{\alpha} \int d^4x \sqrt{-g} (h_{\mu\nu}{}^{;\nu} - \frac{1}{2} g^{\nu\sigma} h_{\nu\sigma;\mu}) g^{\mu\tau} (h_{\tau\lambda}{}^{;\lambda} - \frac{1}{2} g^{\lambda\rho} h_{\lambda\rho;\tau}) \quad (36)$$

where the gauge parameter  $\alpha$  for simplicity is taken positive. The gauge breaking term (36) necessitates the Faddeev-Popov ghost action:

$$S_{FP} = \frac{1}{\sqrt{\alpha}} \int d^4x \sqrt{-g} \bar{\xi}^\mu (\xi_{\mu;\nu}{}^{;\nu} + R_{\mu\nu} \xi^\nu) + O(\kappa). \quad (37)$$

Here the factor  $\frac{1}{\sqrt{\alpha}}$  in front, which usually is disregarded, is of crucial importance for the analysis of quadratic divergences.

The one-loop effective action of quantum gravity is by (30), disregarding for a moment the ghost contribution:

$$\Gamma_{\text{gr}}^{[1]} = -\frac{i}{2} \int_0^\infty \frac{d\tau}{\tau} \int d^4x \left( \frac{1}{2} h^\alpha{}_{\mu\nu}{}^{,\mu\nu}(x, x'; \tau) - \frac{1}{4} h^\alpha{}_{\mu}{}^{,\mu}{}_{,\nu}{}^{,\nu}(x, x'; \tau) \right) \quad (38)$$

where the heat kernel <sup>1</sup>  $h_{\mu\nu,\xi'\eta'}^\alpha(x, x'; \tau)$  is determined by the differential equation according to (35)-(36):

$$i \frac{\partial}{\partial \tau} h_{\mu\nu,\xi'\eta'}^\alpha(x, x'; \tau) + h_{\mu\nu,\xi'\eta'}^\alpha(x, x'; \tau)_{;\sigma}{}^\sigma - \frac{1}{2} X_{\mu\nu}{}^{\lambda\rho} h_{\lambda\rho,\xi'\eta'}^\alpha(x, x'; \tau) \\ - 2R^\lambda{}_{\mu\nu}{}^\rho h_{\lambda\rho,\xi'\eta'}^\alpha(x, x'; \tau) \\ - (1 - \frac{1}{\alpha}) (h_{\mu\lambda,\xi'\eta'}^\alpha(x, x'; \tau)_{;\nu}{}^\lambda + h_{\nu\lambda,\xi'\eta'}^\alpha(x, x'; \tau)_{;\mu}{}^\lambda - g^{\lambda\rho} h_{\lambda\rho,\xi'\eta'}^\alpha(x, x'; \tau)_{;\mu;\nu}) \\ = 0 \quad (39)$$

with the boundary condition:

$$h_{\mu\nu,\xi'\eta'}^\alpha(x, x'; 0) = (g_{\mu\xi'} g_{\nu\eta'} + g_{\nu\xi'} g_{\mu\eta'} - g_{\mu\nu} g_{\xi'\eta'}) \delta(x, x') \quad (40)$$

and where:

$$\frac{1}{2} X_{\mu\nu}{}^{\lambda\rho} = R_{\mu\nu} g^{\lambda\rho} + g_{\mu\nu} \mathcal{G}^{\lambda\rho} - \delta_\mu{}^\lambda \mathcal{G}_\nu{}^\rho - \delta_\nu{}^\lambda \mathcal{G}_\mu{}^\rho + 2R^\lambda{}_{\mu\nu}{}^\rho. \quad (41)$$

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<sup>1</sup>It is somewhat misleading to refer to this quantity as a heat kernel; this requires the proper time variable  $\tau$  to be imaginary whereas it is assumed real here. However, we shall continue to use this name for simplicity.



The ghost contribution to the effective action is by (37):

$$\Gamma_{\text{gh}}^{[1]} = i \int_0^\infty \frac{d\tau}{\tau} \int d^4x h_{\text{gh},\mu}{}^\mu(x, x; \frac{1}{\sqrt{\alpha}}\tau) \quad (42)$$

with:

$$i \frac{\partial}{\partial \tau} h_{\text{gh},\mu,\xi'}(x, x'; \tau) + h_{\text{gh},\mu,\xi'}(x, x'; \tau)_{;\sigma}{}^\sigma + R_\mu{}^\nu h_{\text{gh},\nu,\xi'}(x, x'; \tau) = 0 \quad (43)$$

and:

$$h_{\text{gh},\mu,\xi'}(x, x'; 0) = g_{\mu\xi'} \delta(x, x'). \quad (44)$$

This heat kernel fulfills the following important relation:

$$h_{\text{gh},\mu\xi'}(x, x'; \tau_1 + \tau_2) = \int d^4x'' h_{\text{gh},\mu}{}^{\sigma''}(x, x''; \tau_1) h_{\text{gh},\sigma''\xi'}(x'', x'; \tau_2). \quad (45)$$

by (37). An analogous relation holds for the graviton heat kernel.

One should notice the square root of the gauge parameter  $\alpha$  in (42). Formally this dependence on  $\alpha$  can be removed by a rescaling of the integration variable  $\tau$ . When a lower cut-off is introduced in the integral, however, the quartic and quadratic divergences will depend on  $\alpha$ .

The dependence of the heat kernel  $h_{\mu\nu,\xi'\eta'}^\alpha(x, x'; \tau)$  on the gauge parameter  $\alpha$  is next determined. First it is shown that the heat kernel obeys a Ward identity. From (39) follows:

$$\begin{aligned} & i \frac{\partial}{\partial \tau} (h_{\mu\nu,\xi'\eta'}^\alpha(x, x'; \tau)_{;\mu}{}^\mu - \frac{1}{2} h_{\mu}{}^{\alpha\mu}{}_{,\xi'\eta'}(x, x'; \tau)_{;\nu}{}^\nu) \\ & + \frac{1}{\alpha} (h_{\mu\nu,\xi'\eta'}^\alpha(x, x'; \tau)_{;\mu}{}^\mu - \frac{1}{2} h_{\mu}{}^{\alpha\mu}{}_{,\xi'\eta'}(x, x'; \tau)_{;\sigma}{}^\sigma)_{;\sigma}{}^\sigma \\ & + \frac{1}{\alpha} R_\nu{}^\lambda (h_{\mu\lambda,\xi'\eta'}^\alpha(x, x'; \tau)_{;\mu}{}^\mu - \frac{1}{2} h_{\mu}{}^{\alpha\mu}{}_{,\xi'\eta'}(x, x'; \tau)_{;\lambda}{}^\lambda) \\ & = -\mathcal{G}^{\lambda\rho} (2h_{\lambda\nu,\xi'\eta'}^\alpha(x, x'; \tau)_{;\rho} - h_{\lambda\rho,\xi'\eta'}^\alpha(x, x'; \tau)_{;\nu}) \end{aligned} \quad (46)$$

with the solution:

$$\begin{aligned} & h_{\mu\nu,\xi'\eta'}^\alpha(x, x'; \tau)_{;\mu}{}^\mu - \frac{1}{2} h_{\mu}{}^{\alpha\mu}{}_{,\xi'\eta'}(x, x'; \tau)_{;\nu}{}^\nu \\ & = -h_{\text{gh},\nu,\eta'}(x, x'; \frac{1}{\alpha}\tau)_{;\xi'} - h_{\text{gh},\nu,\xi'}(x, x'; \frac{1}{\alpha}\tau)_{;\eta'} \\ & + i\tau \int d^4x'' \int_0^1 dt h_{\text{gh},\nu}{}^{\sigma''}(x, x''; \frac{1}{\alpha}t\tau) \mathcal{G}^{\omega''\delta''}(x'') (2h_{\omega''\sigma'',\xi'\eta'}^\alpha(x'', x'; (1-t)\tau)_{;\delta''} \\ & - h_{\omega''\delta'',\xi'\eta'}^\alpha(x'', x'; (1-t)\tau)_{;\sigma''}). \end{aligned} \quad (47)$$

From (39) also follows a differential equation for the function  $\alpha \frac{\partial}{\partial \alpha} h_{\mu\nu,\xi'\eta'}^\alpha(x, x'; \tau)$ :

$$\begin{aligned} & i \frac{\partial}{\partial \tau} (\alpha \frac{\partial}{\partial \alpha} h_{\mu\nu,\xi'\eta'}^\alpha(x, x'; \tau)) + \alpha \frac{\partial}{\partial \alpha} h_{\mu\nu,\xi'\eta'}^\alpha(x, x'; \tau)_{;\sigma}{}^\sigma - \frac{1}{2} X_{\mu\nu}{}^{\lambda\rho} \alpha \frac{\partial}{\partial \alpha} h_{\lambda\rho,\xi'\eta'}^\alpha(x, x'; \tau) \\ & - (1 - \frac{1}{\alpha}) (\alpha \frac{\partial}{\partial \alpha} h_{\mu\lambda,\xi'\eta'}^\alpha(x, x'; \tau)_{;\nu}{}^\nu + \alpha \frac{\partial}{\partial \alpha} h_{\nu\lambda,\xi'\eta'}^\alpha(x, x'; \tau)_{;\mu}{}^\mu - g^{\lambda\rho} \alpha \frac{\partial}{\partial \alpha} h_{\lambda\rho,\xi'\eta'}^\alpha(x, x'; \tau)_{;\mu;\nu}) \\ & = \frac{1}{\alpha} (h_{\mu\lambda,\xi'\eta'}^\alpha(x, x'; \tau)_{;\nu}{}^\nu + h_{\nu\lambda,\xi'\eta'}^\alpha(x, x'; \tau)_{;\mu}{}^\mu - g^{\lambda\rho} h_{\lambda\rho,\xi'\eta'}^\alpha(x, x'; \tau)_{;\mu;\nu}). \end{aligned} \quad (48)$$

This equation is solved in the same way as (46); the result is:

$$\begin{aligned}
& \alpha \frac{\partial}{\partial \alpha} h_{\mu\nu, \xi' \eta'}^\alpha(x, x'; \tau) \\
&= \frac{1}{\alpha} i\tau \int d^4 x'' \int_0^1 dt (h_{\mu\nu, \lambda'' \rho''}^\alpha(x, x''; t\tau)_{;\lambda''} - \frac{1}{2} h_{\mu\nu, \lambda''}^{\alpha \lambda''}(x, x''; t\tau)_{;\rho''}) g^{\rho'' \omega''}(x'') \\
& (h_{\sigma'' \omega''}^\alpha(x'', x'; (1-t)\tau)_{;\sigma''} - \frac{1}{2} h_{\sigma''}^{\alpha \sigma''}(x'', x'; (1-t)\tau)_{;\omega''}). \tag{49}
\end{aligned}$$

Using here (47) and disregarding the second term on the right-hand side one gets approximately:

$$\begin{aligned}
& \alpha \frac{\partial}{\partial \alpha} h_{\mu\nu, \xi' \eta'}^\alpha(x, x'; \tau) \simeq -\frac{1}{\alpha} i\tau \int d^4 x'' \int_0^1 dt (h_{\text{gh}\mu\rho''}(x, x''; t\frac{1}{\alpha}\tau)_{;\nu} + h_{\text{gh}\nu\rho''}(x, x''; t\frac{1}{\alpha}\tau)_{;\mu} g^{\rho'' \omega''}(x'') \\
& (h_{\sigma'' \omega''}^\alpha(x'', x'; (1-t)\tau)_{;\sigma''} - \frac{1}{2} h_{\sigma''}^{\alpha \sigma''}(x'', x'; (1-t)\tau)_{;\omega''}). \tag{50}
\end{aligned}$$

Using again (47) and also (45) one gets a further approximation:

$$\begin{aligned}
& \alpha \frac{\partial}{\partial \alpha} h_{\mu\nu, \xi' \eta'}^\alpha(x, x'; \tau) \simeq \frac{1}{\alpha} i\tau ((h_{\text{gh}, \nu, \eta'}(x, x'; \frac{1}{\alpha}\tau))_{;\mu; \xi'} + (h_{\text{gh}, \nu, \xi'}(x, x'; \frac{1}{\alpha}\tau))_{;\mu; \eta'} \\
& + (h_{\text{gh}, \mu, \eta'}(x, x'; \frac{1}{\alpha}\tau))_{;\nu; \xi'} + (h_{\text{gh}, \mu, \xi'}(x, x'; \frac{1}{\alpha}\tau))_{;\nu; \eta'}). \tag{51}
\end{aligned}$$

Combining (49) with (38) and the relation analogous to (45) for the graviton heat kernel the dependence of the effective action of the gauge parameter  $\alpha$  can be found:

$$\begin{aligned}
& \alpha \frac{\partial}{\partial \alpha} \Gamma_{\text{gr}}^{[1]} = \frac{1}{2} \frac{1}{\alpha} \int_0^\infty d\tau \int d^4 x g^{\nu\eta'} (h_{\mu\nu, \xi' \eta'}^\alpha(x, x'; \tau)_{;\mu}^{\xi'} - \frac{1}{2} h_{\mu}^{\alpha\mu}(x, x'; \tau)_{;\nu}^{\xi'}) \\
& - \frac{1}{2} h_{\mu\nu, \xi'}^\alpha(x, x'; \tau)_{;\mu}^{\xi'} + \frac{1}{4} h_{\mu}^{\alpha\mu}(x, x'; \tau)_{;\nu; \eta'}^{\xi'}). \tag{52}
\end{aligned}$$

By means of (47) one gets from (52) two terms that correspond precisely to the two terms of (26):

$$\begin{aligned}
& \alpha \frac{\partial}{\partial \alpha} \Gamma_{\text{gr}, I}^{[1]} = \frac{1}{2} \frac{1}{\alpha} \int_0^\infty d\tau \int d^4 x g^{\mu\eta'} (-h_{\text{gh}, \mu, \eta'}(x, x'; \frac{1}{\alpha}\tau)_{;\xi'}^{\xi'} - h_{\text{gh}, \mu, \xi'}(x, x'; \frac{1}{\alpha}\tau)_{;\eta'}^{\xi'}) \\
& + h_{\text{gh}, \mu, \xi'}(x, x'; \frac{1}{\alpha}\tau)_{;\eta'}^{\xi'}) |_{x' \rightarrow x} \\
& = \frac{1}{2} i \int d^4 x g^{\mu\eta'} \int_0^\infty d\tau \frac{\partial}{\partial \tau} h_{\text{gh}, \mu, \eta'}(x, x; \frac{1}{\alpha}\tau) \tag{53}
\end{aligned}$$

where (43) was used in the last step, and:

$$\begin{aligned}
& \alpha \frac{\partial}{\partial \alpha} \Gamma_{\text{gr}, II}^{[1]} = i \frac{1}{\alpha} \int d^4 x \int d^4 x' \int_0^\infty \tau d\tau \int_0^1 dt h_{\text{gh}}^{\mu\sigma'}(x, x'; t\frac{1}{\alpha}\tau) \mathcal{G}^{\omega'\delta'}(x') \\
& (h_{\omega'\sigma', \mu\nu}^\alpha(x', x; (1-t)\tau)_{;\delta'}^{\nu} - \frac{1}{2} h_{\omega'\sigma', \nu}^{\alpha \nu}(x', x; (1-t)\tau)_{;\delta'; \mu} \\
& - \frac{1}{2} (h_{\omega'\delta', \mu\nu}^\alpha(x', x; (1-t)\tau)_{;\sigma'}^{\nu} - \frac{1}{2} h_{\omega'\delta', \nu}^{\alpha \nu}(x', x; (1-t)\tau)_{;\sigma'; \mu})). \tag{54}
\end{aligned}$$

(53) is an integral of a total derivative and formally vanishes. It is considered in connection with the ghost contribution to the effective action (42) from which one finds:

$$\alpha \frac{\partial}{\partial \alpha} \Gamma_{\text{gh}}^{[1]} = -\frac{1}{2}i \int d^4x \int_0^\infty d\tau \frac{\partial}{\partial \tau} h_{\text{gh},\mu}{}^\mu(x, x; \frac{1}{\sqrt{\alpha}}\tau) \quad (55)$$

that has the same form as (53), with a sign change and a different dependence on  $\alpha$ . It would cancel with (53) by a rescaling of the proper time variable  $\tau$ . Indeed, it is an example of the partial cancellation between the general expressions (26) and (27). However, a careful examination of the quartic and quadratic divergences is necessary here. Using (148) and (149) one gets from (53):

$$\alpha \frac{\partial}{\partial \alpha} \Gamma_{\text{gr},I}^{[1]} \simeq \frac{1}{2} \frac{1}{16\pi^2} \int d^4x \sqrt{-g} \left( -\frac{4\alpha^2}{\tau^2} + \frac{5}{3} R \frac{i\alpha}{\tau} \right) |_{\tau \simeq 0} \quad (56)$$

where a conventional cut-off  $\Lambda$  can be introduced by the substitution

$$\tau \simeq -i \frac{1}{\Lambda^2}. \quad (57)$$

From (55) one gets:

$$\alpha \frac{\partial}{\partial \alpha} \Gamma_{\text{gh}}^{[1]} \simeq -\frac{1}{2} \frac{1}{16\pi^2} \int d^4x \sqrt{-g} \left( -\frac{4\alpha}{\tau^2} + \frac{5}{3} R \frac{i\sqrt{\alpha}}{\tau} \right) |_{\tau \simeq 0} \quad (58)$$

that does not cancel with (56) at general  $\alpha$ .

The expression (54) is also quadratically divergent, and its divergent part is determined by first using the Ward identity (47), disregarding the second term on the right hand side, and also the relation (45):

$$\alpha \frac{\partial}{\partial \alpha} \Gamma_{\text{gr},II}^{[1]} \simeq -i \frac{1}{\alpha} \int d^4x \int_0^\infty \tau d\tau \mathcal{G}^{\omega'\delta'}(x) h_{\text{gh},\sigma'\mu}(x', x; \frac{1}{\alpha}\tau)_{;\omega';\delta'} |_{x' \rightarrow x} g^{\mu\sigma'}. \quad (59)$$

By means of (152) one then obtains from (59):

$$\alpha \frac{\partial}{\partial \alpha} \Gamma_{\text{gr},II}^{[1]} \simeq -2\alpha^2 \frac{i}{16\pi^2} \frac{1}{\tau} |_{\tau \simeq 0} \int d^4x \sqrt{-g} R. \quad (60)$$

The gauge dependence of the one-loop effective action of pure gravity (38) is contained in the expressions (53), which is formally cancelled by the ghost contribution (55), as well as in (54) that vanishes in an Einstein-flat space-time with  $\mathcal{G}^{\mu\nu} = 0$  and where the gauge parameter dependence is expected to be formally removed by the Vilkovisky construction. In the presence of matter fields we expect that the gauge dependent part of the effective action still contains (54), with the replacement  $\mathcal{G}^{\mu\nu} \rightarrow \mathcal{G}^{\mu\nu} - \mathcal{T}^{\mu\nu}$ , where  $\mathcal{T}^{\mu\nu}$  is the energy-momentum tensor of the background matter field.

We then work out the details of the Vilkovisky construction in quantum gravity. This topic has previously been considered in [15], [16]. We here use the proper-time representation of the effective action, such that the formal

cancellation of (29) with (26) and (27) can be investigated on the regularized level in this case.

When the effective action is extended by Vilkovisky and De Witt's method to field configurations where the background field equations are not valid, new terms are introduced in quantum gravity by (22), with:

$$S_{,h_{\mu\nu}} = -\frac{1}{\kappa} \mathcal{G}^{\mu\nu}. \quad (61)$$

In quantum gravity the field metric can be chosen as:

$$G_{h_{\mu\nu}(x), h_{\lambda'\rho'}(x')} = \frac{1}{4} \sqrt{-g} (g^{\mu\lambda'} g^{\nu\rho'} + g^{\nu\lambda'} g^{\mu\rho'} - g^{\mu\nu} g^{\lambda'\rho'}) (x) \delta(x, x'). \quad (62)$$

For gravitational fluctuations  $h_{\mu\nu}$  the transformation (33) determines by means of (16) and (43):

$$N^{\alpha\beta} \rightarrow N^{\xi_\mu(x)\xi_{\nu'}(x')} = i\alpha^k \int_0^\infty d\tau \frac{1}{4\sqrt{-g}} h_{\text{gh},\mu\nu'}(x, x'; \alpha^k \tau) \frac{1}{4\sqrt{-g'}} \quad (63)$$

with  $k$  so far unspecified.

Using also (33) one now finds:

$$\begin{aligned} & \frac{i}{2} S[\phi_0]_{,j} \Delta[\phi_0]^{nm} R^j_{\alpha,n} N^{\alpha\beta} R^k_{\beta} \gamma_{km} \\ & \rightarrow \frac{1}{2} \int d^4x \int d^4y \int d^4z \int d^4w \int d^4u \int d^4t S_{,h_{\mu\nu}(x)} R^{h_{\mu\nu}(x)}_{\xi_\omega(y), h_{\lambda\rho}(z)} \\ & N^{\xi_\omega(y)\xi_\sigma(w)} R^{h_{\xi_\eta(u)}_{\xi_\sigma(w)}} G_{\xi_\eta, \alpha\beta}(u, t) < h_{\alpha\beta}(t) h_{\lambda\rho}(z) > \\ & \simeq -i\alpha^k \int d^4x \int d^4x' \int_0^\infty \tau d\tau \int_0^1 dt h_{\text{gh}, \mu\sigma'}(x, x'; t\alpha^k \tau) \mathcal{G}^{\omega'\delta'}(x') \\ & (h_{\omega'\sigma', \mu\nu}^\alpha(x', x; (1-t)\tau)_{;\delta'}{}^{\nu} - \frac{1}{2} h_{\omega'\sigma', \nu}^\alpha(x', x; (1-t)\tau)_{;\delta'; \mu} \\ & - \frac{1}{2} (h_{\omega'\delta', \mu\nu}^\alpha(x', x; (1-t)\tau)_{;\sigma'}{}^{\nu} - \frac{1}{2} h_{\omega'\delta', \nu}^\alpha(x', x; (1-t)\tau)_{;\sigma'; \mu})) \end{aligned} \quad (64)$$

where the graviton propagator is:

$$< h_{\mu\nu}(x) h_{\xi'\eta'}(x') > = \int_0^\infty d\tau \frac{1}{4\sqrt{-g}} h_{\mu\nu, \xi'\eta'}^\alpha(x, x'; \tau) \frac{1}{4\sqrt{-g'}}. \quad (65)$$

(64) vanishes in the Landau-DeWitt gauge (the limit  $\alpha = 0$ ) by the Ward identity (47), where only the first hand on the right-hand side is kept. Requiring that (64) cancels with (54) fixes  $k$  at  $-1$ ; at other values of  $k$  there is formally still cancellation as seen by introducing  $\tau_1 = t\tau, \tau_2 = (1-t)\tau$  and performing a rescaling of  $\tau_1$ . However, this argument is upset by the quadratic divergences of the two expressions. Evaluating the quadratic divergence of (64) at general  $k$  in the same way as (60) by means of (45) one finds:

$$- \frac{1}{16\pi^2} i\alpha^2 \frac{1 - \alpha^{-2(k+1)}}{1 - \alpha^{k+1}} \frac{1}{\tau} \Big|_{\tau \simeq 0} \int d^4x \sqrt{-g} R \quad (66)$$

that only cancels (60) at  $k = -1$ , showing that the requirement that (64) cancels with (54) also for quadratic divergences is a nontrivial one.

The Christoffel connection in field space is:

$$\begin{aligned} & \Gamma^{h_{\sigma\omega}(x)}_{h_{\mu\nu}(y), h_{\lambda\rho}(z)} \\ &= \frac{1}{4} \left( \delta_{(\sigma\omega)}^{(\mu\nu)} g^{\lambda\rho} + \delta_{(\sigma\omega)}^{(\lambda\rho)} g^{\mu\nu} - \delta_{(\sigma\omega)}^{(\nu\rho)} g^{\mu\lambda} - \delta_{(\sigma\omega)}^{(\nu\lambda)} g^{\mu\rho} - \delta_{(\sigma\omega)}^{(\mu\rho)} g^{\nu\lambda} \right. \\ & \quad \left. - \delta_{(\sigma\omega)}^{(\mu\lambda)} g^{\nu\rho} + g^{(\mu\nu)(\lambda\rho)} g_{\sigma\omega} - \frac{1}{2} g^{\mu\nu} g^{\lambda\rho} g_{\sigma\omega} \right) (y) \delta(x, y) \delta(y, z) \end{aligned} \quad (67)$$

with:

$$\delta_{(\mu\nu)}^{(\sigma\omega)} = \frac{1}{2} (\delta_\mu^\sigma \delta_\nu^\omega + \delta_\nu^\sigma \delta_\mu^\omega), \quad g^{(\mu\nu)(\lambda\rho)} = \frac{1}{2} (g^{\mu\lambda} g^{\nu\rho} + g^{\mu\rho} g^{\nu\lambda}). \quad (68)$$

From (63) follows that the projection operator  $\Pi^m_n$  defined in (17) in this case is:

$$\begin{aligned} & \Pi^m_n \rightarrow \Pi^{h_{\mu\nu}(x)}_{h_{\lambda\rho}(y)} = \delta_{(\mu\nu)}^{(\lambda\rho)} \delta(x, y) \\ & - i^4 \sqrt{-g} \frac{1}{\sqrt{\alpha}} < (\xi_{\mu;\nu} + \xi_{\nu;\mu})(x) \frac{1}{2} (\bar{\xi}^{\lambda;\rho} + \bar{\xi}^{\rho;\lambda} - g^{\lambda\rho} \bar{\xi}_{\sigma;\sigma})(y) >^4 \sqrt{-g} \end{aligned} \quad (69)$$

where the ghost propagator is:

$$< \xi_\mu(x) \bar{\xi}_\nu(y) > = \frac{1}{\sqrt{\alpha}} \int_0^\infty d\tau \frac{1}{4\sqrt{-g}} h_{\text{gh},\mu,\xi'}(x, x'; \frac{1}{\alpha}\tau) \frac{1}{4\sqrt{-g'}}. \quad (70)$$

Applying this projection operator upon the graviton propagator one obtains the graviton propagator in the Landau-DeWitt gauge. From (22) one gets by (67):

$$\begin{aligned} & -\frac{i}{2} S[\phi_0]_{,m} \Gamma^m_{kl}[\phi_0] \Pi^k_r \Delta[\phi_0]^{rs} \Pi^l_s \\ & \rightarrow \frac{1}{8} \int d^4x \sqrt{-g} \mathcal{G}^{\sigma\omega}(x) < h_{\mu\nu}(x) h_{\lambda\rho}(x) > \\ & \left( \delta_{(\sigma\omega)}^{(\mu\nu)} g^{\lambda\rho} + \delta_{(\sigma\omega)}^{(\lambda\rho)} g^{\mu\nu} - \delta_{(\sigma\omega)}^{(\nu\rho)} g^{\mu\lambda} - \delta_{(\sigma\omega)}^{(\nu\lambda)} g^{\mu\rho} - \delta_{(\sigma\omega)}^{(\mu\rho)} g^{\nu\lambda} - \delta_{(\sigma\omega)}^{(\mu\lambda)} g^{\nu\rho} \right. \\ & \quad \left. + g^{(\mu\nu)(\lambda\rho)} g_{\sigma\omega} - \frac{1}{2} g^{\mu\nu} g^{\lambda\rho} g_{\sigma\omega} \right) \end{aligned} \quad (71)$$

with the graviton propagator in the Landau-DeWitt gauge. This expression has no quadratic divergence in four dimensions by (153). There is no contribution to the effective action in this case from the final term in (22).

## 4 The Maxwell-Einstein system

### 4.1 Maxwell field in a curved background

The Maxwell field  $A_\mu$  has the action:

$$S_M = \int d^4x \sqrt{-g} g^{\mu\lambda} g^{\nu\rho} \left( -\frac{1}{4} F_{\mu\nu} F_{\lambda\rho} \right); \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (72)$$

Here  $A_\mu$  can be considered a covariant vector with the following transformation rule under infinitesimal coordinate transformations:

$$\delta A_\mu = \kappa(\xi^\lambda{}_{;\mu} A_\lambda + \xi^\lambda A_{\mu;\lambda}) + O(\kappa^2). \quad (73)$$

(72) gets by the splitting (32) the additional term:

$$S_M^{(1)} = \frac{1}{\kappa} \int d^4x \sqrt{-g} h_{\mu\nu} T^{\mu\nu} \quad (74)$$

where the energy-momentum tensor  $T^{\mu\nu}$  is:

$$T^{\mu\nu} = \frac{\kappa^2}{2} (F^{\mu\lambda} F^\nu{}_\lambda - \frac{1}{4} g^{\mu\nu} F^{\lambda\rho} F_{\lambda\rho}). \quad (75)$$

At second order in  $\kappa$  one gets from (72):

$$\begin{aligned} S_M^{(2)} = & \kappa^2 \int d^4x \sqrt{-g} h_{\omega\tau} h_{\iota\sigma} \left( \frac{1}{8} g^{\omega\tau} (F^\iota{}_\lambda F^{\sigma\lambda} - \frac{1}{4} g^{\iota\sigma} F^{\mu\nu} F_{\mu\nu}) + \frac{1}{16} g^{\omega\iota} g^{\sigma\tau} F^{\mu\nu} F_{\mu\nu} \right. \\ & \left. + \frac{1}{8} g^{\omega\tau} F^\iota{}_\lambda F^{\sigma\lambda} - \frac{1}{4} F^{\omega\iota} F^{\tau\sigma} - \frac{1}{2} g^{\sigma\omega} F^\iota{}_\lambda F^{\tau\lambda} \right). \end{aligned} \quad (76)$$

The gauge breaking action of the Maxwell field is:

$$S_{M,GB} = \int d^4x \sqrt{-g} \left( -\frac{1}{2} \frac{1}{\beta} (A_\mu{}^{;\mu})^2 \right) \quad (77)$$

with the gauge parameter  $\beta > 0$ , and the corresponding ghost action:

$$S_{M,FP} = \frac{1}{\sqrt{\beta}} \int d^4x \sqrt{-g} \bar{c} (c_{;\mu} + \kappa(\xi^\lambda{}_{;\mu} A_\lambda + \xi^\lambda A_{\mu;\lambda}))^{;\mu} \quad (78)$$

with  $c$  a scalar ghost and  $\bar{c}$  the corresponding antighost.

The photon heat kernel  $h_{\mu,\xi'}^\beta(x, x'; \tau)$  is defined by:

$$\begin{aligned} & i \frac{\partial}{\partial \tau} h_{\mu,\xi'}^\beta(x, x'; \tau) + h_{\mu,\xi'}^\beta(x, x'; \tau)_{;\sigma}{}^{;\sigma} \\ & - R_\mu{}^\nu h_{\nu,\xi'}^\beta(x, x'; \tau) - (1 - \frac{1}{\beta}) h_{\nu,\xi'}^\beta(x, x'; \tau)_{;\nu}{}^{;\mu} = 0 \end{aligned} \quad (79)$$

where the boundary condition is:

$$h_{\mu,\xi'}^\beta(x, x'; 0) = g_{\mu\xi'} \delta(x, x'). \quad (80)$$

Also the scalar heat kernel  $h(x, x'; \tau)$  is defined by:

$$i \frac{\partial}{\partial \tau} h(x, x'; \tau) + h(x, x'; \tau)_{;\sigma}{}^{;\sigma} = 0; h(x, x'; 0) = \delta(x, x'). \quad (81)$$

From (79) follows:

$$i \frac{\partial}{\partial \tau} h_{\mu,\xi'}^\beta(x, x'; \tau)_{;\mu}{}^{;\mu} + \frac{1}{\beta} h_{\mu,\xi'}^\beta(x, x'; \tau)_{;\mu}{}^{;\mu}{}_{;\sigma}{}^{;\sigma} = 0 \quad (82)$$

the solution of which is the following Ward identity:

$$h_{\mu,\xi'}^\beta(x, x'; \tau);^\mu = -h(x, x'; \frac{1}{\beta}\tau);_{\xi'} \quad (83)$$

obtained by comparison of (81) and (82) and by taking the boundary conditions into account. Also (83) combined with (81) imply:

$$h_{\mu,\xi'}^\beta(x, x'; \tau) = h_{\mu,\xi'}(x, x'; \tau) - i \int_\tau^{\frac{1}{\beta}\tau} d\tau' h(x, x'; \tau');_{\mu;\xi'} \quad (84)$$

where  $h_{\mu,\xi'}(x, x'; \tau)$  is the heat kernel for  $\beta = 1$ .

The Maxwell field one-loop action in an arbitrary curved background is:

$$\Gamma_M^{[1]} = \frac{i}{2} \int_0^\infty \frac{d\tau}{\tau} \int d^4x h_{\mu}^{\beta,\mu}(x, x; \tau) \quad (85)$$

with the gauge dependence according to (84):

$$\beta \frac{\partial}{\partial \beta} \Gamma_M^{[1]} = \frac{1}{2} i \int_0^\infty d\tau \frac{\partial}{\partial \tau} h(x, x; \frac{1}{\beta}\tau). \quad (86)$$

Also, the ghost action is here:

$$\Gamma_{M,\text{gh}}^{[1]} = i \int_0^\infty \frac{d\tau}{\tau} \int d^4x h(x, x; \frac{1}{\sqrt{\beta}}\tau) \quad (87)$$

with:

$$\beta \frac{\partial}{\partial \beta} \Gamma_{M,\text{gh}}^{[1]} = -\frac{1}{2} i \int d^4x \int_0^\infty d\tau \frac{\partial}{\partial \tau} h(x, x; \frac{1}{\sqrt{\beta}}\tau) \quad (88)$$

that formally cancels with (86) by a rescaling of  $\tau$  but where the cancellation is upset by divergent terms. The determination of these divergent terms is carried out by (148) and (150) in the same way as for (56) and (58), and one gets from (86):

$$\beta \frac{\partial}{\partial \beta} \Gamma_M^{[1]} \simeq \frac{1}{2} \frac{1}{16\pi^2} \left( -\frac{\beta^2}{\tau^2} + \frac{1}{6} R \frac{i\beta}{\tau} \right) |_{\tau \simeq 0} \quad (89)$$

and from (88):

$$\beta \frac{\partial}{\partial \beta} \Gamma_{M,\text{gh}}^{[1]} \simeq -\frac{1}{2} \frac{1}{16\pi^2} \left( -\frac{\beta}{\tau^2} + \frac{1}{6} R \frac{i\sqrt{\beta}}{\tau} \right) |_{\tau \simeq 0} \quad (90)$$

that do not cancel at general values of  $\beta$ .

## 4.2 Gauge dependence at order $\kappa^2$ of Maxwell-Einstein theory

In the Maxwell action  $S_M$  a background field  $\mathcal{A}_\mu$  is introduced, with the corresponding field strength  $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$ , with:

$$\mathcal{F}_{\mu\nu};^\mu = 0 \quad (91)$$

and the field  $A_\mu$  is split according to:

$$A_\mu \rightarrow \mathcal{A}_\mu + A_\mu \quad (92)$$

with  $A_\mu$  still the quantum field. The two-point correlation function is:

$$\langle A_\mu(x) A_{\lambda'}(x') \rangle = \int_0^\infty d\tau \frac{1}{4\sqrt{-g}} h_{\mu\lambda'}^\beta(x, x'; \tau) \frac{1}{4\sqrt{-g'}}. \quad (93)$$

The identity (31) makes it straightforward to carry out a perturbation expansion of the effective action in the proper-time representation. At second order in  $\kappa$  there is a two-point function term by (74):

$$\begin{aligned} \Gamma_{\mathbf{EM},I}^{[1]} = & \frac{i}{2} \kappa^2 \int d^4x \int d^4x' \int_0^\infty \tau d\tau \int_0^1 dt h_{\mu\nu,\xi'\eta'}^\alpha(x, x'; t\tau) (h_{\rho\delta'}^\beta(x, x'; (1-t)\tau)_{;\lambda;\gamma'} \\ & - h_{\rho\gamma'}^\beta(x, x'; (1-t)\tau)_{;\lambda;\delta'} - h_{\lambda\beta'}^\beta(x, x'; (1-t)\tau)_{;\rho;\alpha'} + h_{\lambda\gamma'}^\beta(x, x'; (1-t)\tau)_{;\rho;\delta'}) \\ & (g^{\nu\lambda} \mathcal{F}^{\mu\rho} - \frac{1}{4} g^{\mu\nu} \mathcal{F}^{\lambda\rho})(x) (g^{\eta'\gamma'} \mathcal{F}^{\xi'\delta'} - \frac{1}{4} g^{\xi'\eta'} \mathcal{F}^{\gamma'\delta'})(x'). \end{aligned} \quad (94)$$

This expression has a quadratic divergence at  $\alpha = 1$  by (154):

$$\Gamma_{\mathbf{EM},I}^{[1]} \simeq -\frac{3}{2} \frac{i}{16\pi^2} \frac{1}{\tau} \Big|_{\tau \simeq 0} \kappa^2 \int d^4x \sqrt{-g} \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu}(x). \quad (95)$$

Also there is to this order a tadpole term by (76):

$$\begin{aligned} \Gamma_{\mathbf{EM},II}^{[1]} = & \kappa^2 \int d^4x \int_0^\infty d\tau h_{\omega\tau,\iota\sigma}^\alpha(x, x; \tau) \left( \frac{1}{8} g^{\omega\tau} (\mathcal{F}^\iota{}_\lambda \mathcal{F}^{\sigma\lambda} - \frac{1}{4} g^{\iota\sigma} \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu}) \right. \\ & \left. + \frac{1}{16} g^{\omega\iota} g^{\sigma\tau} \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu} + \frac{1}{8} g^{\omega\tau} \mathcal{F}^\iota{}_\lambda \mathcal{F}^{\sigma\lambda} - \frac{1}{4} \mathcal{F}^{\omega\iota} \mathcal{F}^{\tau\sigma} - \frac{1}{2} g^{\sigma\omega} \mathcal{F}^\iota{}_\lambda \mathcal{F}^{\tau\lambda} \right) \end{aligned} \quad (96)$$

with the quadratically divergent part at  $\alpha = 1$  by (153):

$$\Gamma_{\mathbf{EM},II}^{[1]} \simeq \frac{3}{4} \frac{i}{16\pi^2} \frac{1}{\tau} \Big|_{\tau \simeq 0} \kappa^2 \int d^4x \sqrt{-g} \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu}(x). \quad (97)$$

(94) does not depend on the gauge parameter  $\beta$ , and the dependence on  $\alpha$  is in the lowest approximation found from (50):

$$\begin{aligned} \alpha \frac{\partial}{\partial \alpha} \Gamma_{\mathbf{EM},I}^{[1]} \simeq & -\frac{1}{2} \frac{1}{\alpha} \kappa^2 \int d^4x \int d^4x' \int d^4x'' \int_0^\infty \tau^2 d\tau \int_0^1 dt du dv \delta(1-t-u-v) \\ & h_{\text{gh}\mu\nu''}(x, x''; t \frac{1}{\alpha} \tau) g^{v''\omega''}(x'')(h_{\sigma''\omega'',\xi'\eta'}^\alpha(x'', x'; u\tau)_{;\sigma''} - \frac{1}{2} h_{\sigma''}^{\alpha\sigma''}{}_{;\xi'\eta'}(x'', x'; u\tau)_{;\omega''}) \\ & (h_{\rho\delta'}^\beta(x, x'; v\tau)_{;\lambda;\gamma'} - h_{\rho\gamma'}^\beta(x, x'; v\tau)_{;\lambda;\delta'} - h_{\lambda\delta'}^\beta(x, x'; v\tau)_{;\rho;\gamma'} + h_{\lambda\gamma'}^\beta(x, x'; v\tau)_{;\rho;\delta'}) \\ & \mathcal{F}^{\mu\rho}(x) (\mathcal{F}^{\xi'\delta'} g^{\eta'\gamma'} - \frac{1}{4} g^{\xi'\eta'} \mathcal{F}^{\gamma'\delta'})(x'). \end{aligned} \quad (98)$$

Using (79) in connection with:

$$h_{\lambda\beta'}(x, x'; \tau)_{;\rho;\alpha'}^\lambda - h_{\lambda\alpha'}(x, x'; \tau)_{;\rho;\beta'}^\lambda = 0, \quad (99)$$



following from (83), one gets from (98) two terms:

$$\begin{aligned}
\alpha \frac{\partial}{\partial \alpha} \Gamma_{\mathbf{EM},I}^{[1]} &\rightarrow \frac{1}{2} \frac{1}{\alpha} i \kappa^2 \int d^4 x \int d^4 x' \int d^4 x'' \int_0^\infty d\tau \frac{\partial}{\partial \tau} \tau^2 \int_0^1 dt du dv \delta(1-t-u-v) \\
&h_{\text{gh}\mu\nu''}(x, x''; t \frac{1}{\alpha} \tau) g^{v''\omega''}(x'')(h_{\sigma''\omega'',\xi'\eta'}^\alpha(x'', x'; u\tau)_{;\sigma''} - \frac{1}{2} h_{\sigma''}^{\alpha\sigma''}{}_{,\xi'\eta'}(x'', x'; u\tau)_{;\omega''}) \\
&(h_{\rho\delta'}^\beta(x, x'; v\tau)_{;\gamma'} - h_{\rho\gamma'}^\beta(x, x'; v\tau)_{;\delta'}) \mathcal{F}^{\mu\rho}(x) (\mathcal{F}^{\xi'\delta'} g^{\eta'\gamma'} - \frac{1}{4} g^{\xi'\eta'} \mathcal{F}^{\gamma'\delta'})(x') \quad (100)
\end{aligned}$$

and also, using the Bianchi identity and the field equation of the background gauge field:

$$\begin{aligned}
\alpha \frac{\partial}{\partial \alpha} \Gamma_{\mathbf{EM},I}^{[1]} &\rightarrow -\frac{1}{2} \frac{1}{\alpha} i \kappa^2 \int d^4 x \int d^4 x' \int_0^\infty \tau d\tau \int_0^1 dt h_{\text{gh}\mu\nu'}(x, x'; t \frac{1}{\alpha} \tau)_{;\gamma} g^{v'\omega'}(x') \\
&(h_{\sigma'\omega',\xi\eta}^\alpha(x', x; (1-t)\tau)_{;\sigma'} - \frac{1}{2} h_{\sigma'}^{\alpha\sigma'}{}_{,\xi\eta}(x', x; (1-t)\tau)_{;\omega'}) \\
&g_{\rho\delta} \mathcal{F}^{\mu\rho}(x) (\mathcal{F}^{\xi\delta} g^{\eta\gamma} - \mathcal{F}^{\xi\gamma} g^{\eta\delta} - \frac{1}{2} g^{\xi\eta} \mathcal{F}^{\gamma\delta})(x) \\
&-\frac{1}{4} \frac{1}{\alpha} i \kappa^2 \int d^4 x \int d^4 x' \int_0^\infty \tau d\tau \int_0^1 dt h_{\text{gh}\mu\nu'}(x, x'; t \frac{1}{\alpha} \tau) g^{v'\omega'}(x') \\
&(h_{\sigma'\omega',\xi\eta}^\alpha(x', x; (1-t)\tau)_{;\sigma'} - \frac{1}{2} h_{\sigma'}^{\alpha\sigma'}{}_{,\xi\eta}(x', x; (1-t)\tau)_{;\omega'}) \\
&(\mathcal{F}^{\xi\rho} \mathcal{F}^{\eta}{}_{\rho} - \frac{1}{4} g^{\xi\eta} \mathcal{F}^{\rho\lambda} \mathcal{F}_{\rho\lambda})_{;\mu}(x). \quad (101)
\end{aligned}$$

Turning to (96) one gets by (50):

$$\begin{aligned}
\alpha \frac{\partial}{\partial \alpha} \Gamma_{\mathbf{EM},II}^{[1]} &\simeq -\frac{1}{\alpha} i \kappa^2 \int d^4 x d^4 x' \int_0^\infty \tau d\tau \int_0^1 dt h_{\text{gh},\omega}{}^{\rho'}(x, x'; t \frac{1}{\alpha} \tau)_{;v} \\
&(h_{\sigma'\rho',\xi\eta}^\alpha(x', x; (1-t)\tau)_{;\sigma'} - \frac{1}{2} h_{\sigma'}^{\alpha\sigma'}{}_{,\xi\eta}(x', x; (1-t)\tau)_{;\rho'}) \\
&(\frac{1}{4} g^{\omega\nu} (\mathcal{F}^\xi{}_\lambda \mathcal{F}^{\eta\lambda} - \frac{1}{4} g^{\xi\eta} \mathcal{F}^{\lambda\epsilon} \mathcal{F}_{\lambda\epsilon}) + \frac{1}{8} g^{\omega\xi} g^{\nu\eta} \mathcal{F}^{\lambda\epsilon} \mathcal{F}_{\lambda\epsilon} + \frac{1}{4} g^{\xi\eta} \mathcal{F}^\omega{}_\lambda \mathcal{F}^{\nu\lambda} - \frac{1}{2} \mathcal{F}^{\omega\xi} \mathcal{F}^{\nu\eta} \\
&-\frac{1}{2} g^{\omega\xi} \mathcal{F}^\nu{}_\lambda \mathcal{F}^{\eta\lambda} - \frac{1}{2} g^{\nu\xi} \mathcal{F}^\omega{}_\lambda \mathcal{F}^{\eta\lambda})(x) \quad (102)
\end{aligned}$$

and adding (101) and (102) to (54) one finally gets:

$$\begin{aligned}
\alpha \frac{\partial}{\partial \alpha} \Gamma^{[1]} &\rightarrow i \frac{1}{\alpha} \int d^4 x \int d^4 x' \int_0^\infty \tau d\tau \int_0^1 dt h_{\text{gh},\omega}{}^{\mu\sigma'}(x, x'; t \frac{1}{\alpha} \tau) (\mathcal{G}^{\omega'\delta'} - \mathcal{T}^{\omega'\delta'})(x') \\
&(h_{\omega'\sigma',\mu\nu}^\alpha(x', x; (1-t)\tau)_{;\delta'}{}^{;\nu} - \frac{1}{2} h_{\omega'\sigma',\nu}^{\alpha\sigma'}{}^{;\nu}(x', x; (1-t)\tau)_{;\delta';\mu} \\
&-\frac{1}{2} (h_{\omega'\delta',\mu\nu}^\alpha(x', x; (1-t)\tau)_{;\sigma'}{}^{;\nu} - \frac{1}{2} h_{\omega'\delta',\nu}^{\alpha\sigma'}{}^{;\nu}(x', x; (1-t)\tau)_{;\sigma';\mu})) \quad (103)
\end{aligned}$$

with  $\mathcal{T}^{\mu\nu}$  the background gauge field energy-momentum tensor, i. e. the Einstein tensor  $\mathcal{G}^{\mu\nu}$  in (54) gets the additional term  $-\mathcal{T}^{\mu\nu}$  when the gravitational field is coupled to an Abelian gauge field. It has been verified that this

conclusion also holds for a non-Abelian gauge field, but the proof is not included in this article. The quadratic divergence of (103) is still given by (60) because the trace of the Maxwell field energy-momentum tensor vanishes in four dimensions.

Finding the quadratic divergence of (100) one uses the Ward identity (47), keeping only the first time on the right hand side, in connection with (45), obtaining by a partial integration:

$$\begin{aligned} & \frac{1}{2} \frac{1}{\alpha} i \kappa^2 \int d^4x \int d^4x' \int_0^\infty d\tau \frac{\partial}{\partial \tau} \tau^2 \int_0^1 t dt h_{\text{gh}\mu\xi'}(x, x'; t \frac{1}{\alpha} \tau) \\ & (h_{\rho\delta'}^\beta(x, x'; (1-t)\tau)_{;\gamma';\eta'} - h_{\rho\gamma'}^\beta(x, x'; (1-t)\tau)_{;\delta';\eta'}) \\ & \mathcal{F}^{\mu\rho}(x) (\mathcal{F}^{\xi'\delta'} g^{\eta'\gamma'} + \mathcal{F}^{\eta'\delta'} g^{\xi'\gamma'} - \frac{1}{2} g^{\xi'\eta'} \mathcal{F}^{\gamma'\delta'})(x'). \end{aligned} \quad (104)$$

(104) is evaluated by (155) and contains the quadratic divergence:

$$- \frac{3}{8} \alpha \frac{i}{16\pi^2} \frac{1}{\tau} \Big|_{\tau \rightarrow 0} \kappa^2 \int d^4x \sqrt{-g} \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu}(x). \quad (105)$$

Inserting (105) in (100), integrating and adding (95) and (97) one obtains:

$$- \frac{3}{8} (1 + \alpha) \frac{i}{16\pi^2} \frac{1}{\tau} \Big|_{\tau \rightarrow 0} \kappa^2 \int d^4x \sqrt{-g} \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu}(x) \quad (106)$$

in agreement with Toms [9] when the proper time  $\tau$  is converted to a temperature  $T$  by taking it imaginary. However, it should be kept in mind that the expression (100), which is responsible for the gauge parameter dependence of (106), contains a total derivative in the proper time integral and thus should be disregarded formally.

### 4.3 Vilkovisky's construction in Maxwell-Einstein theory

In Maxwell-Einstein theory (61) is replaced by:

$$S_{,h_{\mu\nu}} = -\frac{1}{\kappa} (\mathcal{G}^{\mu\nu} - \mathcal{T}^{\mu\nu}). \quad (107)$$

In (64) and (71) one thus has to carry out the replacement  $\mathcal{G}^{\mu\nu} \rightarrow \mathcal{G}^{\mu\nu} - \mathcal{T}^{\mu\nu}$ , and adding (64) after this replacement to (103) with the parameter  $k$  fixed at  $-1$  one removes the dependence on the gauge parameter  $\alpha$ . Thus the Vilkovisky construction of quantum gravity is sufficient to remove the gauge dependence also of the full Einstein-Maxwell system in lowest order, without additional modifications. The vanishing of the quadratic divergence of (71) persists after the replacement, and the gauge parameter dependence of (106) is not eliminated through the Vilkovisky construction.

For the Maxwell field one has:

$$R^{A_\mu(x)}{}_{c(y)} \simeq \delta(x, y)_{,\mu}, \quad (108)$$

and the field metric is:

$$G_{A_\mu(x)A_\nu(y)} = \sqrt{-g}g^{\mu\nu}(x)\delta(x, y). \quad (109)$$

The projection operator  $\Pi^m_n$  corresponding to the Maxwell field is in lowest order, cp. (69):

$$\Pi_{\rho\mu'}(x, x') = \Pi_{A^\rho x A^{\mu'}(x')} = g_{\rho\alpha'}\delta(x, x') - i^4\sqrt{-g}\frac{1}{\sqrt{\beta}} < c(x)_{,\rho}\bar{c}(x')_{,\mu'} > \sqrt{-g'} \quad (110)$$

with the ghost propagator expressed in terms of the scalar heat kernel defined in (81):

$$< c(x)\bar{c}(x') > = \frac{1}{\sqrt{\beta}} \int_0^\infty d\tau \frac{1}{4\sqrt{-g}} h(x, x'; \frac{1}{\beta}\tau) \frac{1}{4\sqrt{-g'}}. \quad (111)$$

This operator projects from the photon propagator in an arbitrary gauge the transverse photon propagator. From (72) and (77) follows for the two-point function:

$$< F_{\lambda\rho};^\lambda(x)A_{\mu'}(x') > = i\frac{1}{4\sqrt{-g}}\Pi_{\rho\mu'}(x, x')\frac{1}{4\sqrt{-g'}}. \quad (112)$$

The Christoffel connection components are:

$$\Gamma^{g\rho\sigma(x)}_{A_\mu(y)A_\nu(z)} = \kappa^2\delta_{(\sigma\omega)}^{(\mu\nu)}\delta(x, y)\delta(y, z). \quad (113)$$

From (22) one thus gets the connection coupling term in the effective action:

$$\frac{1}{2} \int d^4x \sqrt{-g} (\mathcal{G}^{\mu\nu} - \mathcal{T}^{\mu\nu})(x) < A_\mu(x)A_\nu(x) > \quad (114)$$

with a transverse photon propagator, and with the quadratic divergence:

$$\frac{3}{2} \frac{i}{16\pi^2} \frac{1}{\tau} \Big|_{\tau \simeq 0} \int d^4x \sqrt{-g} R \quad (115)$$

with  $R$  the scalar curvature and with no contribution from the background gauge field.

## 4.4 General gauge fixing

The gauge breaking action (36) can be generalized to:

$$S_{GB} = -\frac{1}{2} \frac{1}{\alpha} \int d^4x \sqrt{-g} g^{\mu\nu} \chi_\mu \chi_\nu \quad (116)$$

with:

$$\chi_\mu = h_{\mu\nu};^\nu - \frac{1}{2} g^{\nu\sigma} h_{\nu\sigma;\mu} + \kappa(\omega_1 \mathcal{A}_\mu A^\lambda_{;\lambda} + \omega_2 \mathcal{F}_{\lambda\mu} A^\lambda) \quad (117)$$

where  $\omega_1$  and  $\omega_2$  are new gauge parameters. This gives rise to new couplings:

$$-\frac{1}{\alpha} \kappa \int d^4x \sqrt{-g} g^{\mu\nu} (h_{\mu\lambda};^\lambda - \frac{1}{2} h^\lambda_{\lambda;\mu}) (\omega_1 \mathcal{A}_\nu A^\rho_{;\rho} + \omega_2 \mathcal{F}_{\rho\nu} A^\rho) \quad (118)$$

and:

$$-\frac{1}{2}\frac{1}{\alpha}\kappa^2 \int d^4x \sqrt{-g} g^{\mu\nu} (\omega_1 \mathcal{A}_\mu A^\lambda{}_{;\lambda} + \omega_2 \mathcal{F}_{\lambda\mu} A^\lambda) (\omega_1 \mathcal{A}_\nu A^\rho{}_{;\rho} + \omega_2 \mathcal{F}_{\rho\nu} A^\rho). \quad (119)$$

The corresponding ghost action replacing (37) and (78) is, keeping only terms relevant at one-loop order:

$$\begin{aligned} S_{FP} = & \frac{1}{\sqrt{\alpha}} \int d^4x \sqrt{-g} \bar{\xi}^\mu \left( \xi_{\mu;\nu}{}^\nu + R_{\mu\nu} \xi^\nu + \kappa (\omega_1 \mathcal{A}_\mu c_{;\kappa}{}^\kappa + \omega_2 \mathcal{F}_{\lambda\mu} c_{;\kappa}{}^\lambda) \right. \\ & + \kappa^2 (\omega_1 \mathcal{A}_\mu ((\xi^\lambda \mathcal{A}_\lambda)_{;\rho} + \xi^\lambda \mathcal{F}_{\lambda\rho})_{;\rho} + \omega_2 \mathcal{F}_{\rho\mu} ((\xi^\lambda \mathcal{A}_\lambda)_{;\rho} + \xi^\lambda \mathcal{F}_{\lambda\rho})) \\ & \left. + \frac{1}{\sqrt{\beta}} \int d^4x \sqrt{-g} \bar{c} (c_{;\mu} + \kappa ((\xi^\lambda \mathcal{A}_\lambda)_{;\mu} + \xi^\lambda \mathcal{F}_{\lambda\mu}))_{;\mu} \right). \end{aligned} \quad (120)$$

A new one-loop term in the effective action of order  $\kappa^2$  is by (118):

$$\begin{aligned} & i \frac{1}{2} \frac{1}{\alpha^2} \kappa^2 \int d^4x \sqrt{-g} \int d^4x' \sqrt{-g'} < (h_{\mu\nu}{}_{;\nu}{}^\nu - \frac{1}{2} h^\nu{}_{\nu;\mu})(x) (h_{\lambda'\rho'}{}_{;\rho'}{}^{\rho'} - \frac{1}{2} h^{\rho'}{}_{\rho';\lambda'})(x') > \\ & < (\omega_1 \mathcal{A}^\mu A^\sigma{}_{;\sigma} + \omega_2 \mathcal{F}^{\sigma\mu} A_\sigma)(x) (\omega_1 \mathcal{A}^{\lambda'} A^{\gamma'}{}_{;\gamma'} + \omega_2 \mathcal{F}^{\gamma'\lambda'} A_{\gamma'})(x') >. \end{aligned} \quad (121)$$

(121) is expressed in the proper time representation and the Ward identity (47) is applied, disregarding the last term containing the Einstein tensor. Then (121) is:

$$\begin{aligned} & -i \frac{1}{2} \frac{1}{\alpha^2} \kappa^2 \int d^4x \int d^4x' \int_0^\infty \tau d\tau \int_0^1 dt (h_{\text{gh},\lambda'\mu}(x', x; \frac{1}{\alpha} t\tau)_{;\rho'}{}^{\rho'} + R_{\lambda'\rho'} h_{\text{gh},\mu}{}^{\rho'}(x', x; \frac{1}{\alpha} t\tau)) \\ & (\omega_1^2 \mathcal{A}^\mu(x) \mathcal{A}^{\lambda'}(x') h^{\beta,\sigma\gamma'}(x, x'; (1-t)\tau)_{;\sigma;\gamma'} + \omega_2^2 \mathcal{F}^{\sigma\mu}(x) \mathcal{F}^{\gamma'\lambda'}(x') h^{\beta}{}_{\sigma\gamma'}(x, x'; (1-t)\tau) \\ & + 2\omega_1 \omega_2 \mathcal{A}^\mu(x) \mathcal{F}^{\gamma'\lambda'}(x') h^{\beta}{}_{\sigma\gamma'}(x, x'; (1-t)\tau)_{;\sigma}) \end{aligned} \quad (122)$$

containing two terms by (43)-(44):

$$\begin{aligned} & -\frac{1}{2} \frac{1}{\alpha} \kappa^2 \int d^4x \int d^4x' \int_0^\infty d\tau \frac{\partial}{\partial \tau} \tau \int_0^1 dt h_{\text{gh},\lambda'\mu}(x', x; \frac{1}{\alpha} t\tau) \\ & (\omega_1^2 \mathcal{A}^\mu(x) \mathcal{A}^{\lambda'}(x') h^{\beta,\sigma\gamma'}(x, x'; (1-t)\tau)_{;\sigma;\gamma'} + \omega_2^2 \mathcal{F}^{\sigma\mu}(x) \mathcal{F}^{\gamma'\lambda'}(x') h^{\beta}{}_{\sigma\gamma'}(x, x'; (1-t)\tau) \\ & + 2\omega_1 \omega_2 \mathcal{A}^\mu(x) \mathcal{F}^{\gamma'\lambda'}(x') h^{\beta}{}_{\sigma\gamma'}(x, x'; (1-t)\tau)_{;\sigma}) \end{aligned} \quad (123)$$

and:

$$\frac{1}{2} \frac{1}{\alpha} \kappa^2 \int d^4x \sqrt{-g} g^{\mu\nu} < (\omega_1 \mathcal{A}_\mu A^\lambda{}_{;\lambda} + \omega_2 \mathcal{F}_{\lambda\mu} A^\lambda)(x) (\omega_1 \mathcal{A}_\nu A^\rho{}_{;\rho} + \omega_2 \mathcal{F}_{\rho\nu} A^\rho)(x) >. \quad (124)$$

(123) contains an integral of a total derivative in the proper-time variable  $\tau$  in the same way as (100) and only has quartical and quadratically divergent terms, and it should be disregarded formally; its gauge parameter dependence is not removed through the Vilkovisky construction. Also we get from (119):

$$-\frac{1}{2} \frac{1}{\alpha} \kappa^2 \int d^4x \sqrt{-g} g^{\mu\nu} < (\omega_1 \mathcal{A}_\mu A^\lambda{}_{;\lambda} + \omega_2 \mathcal{F}_{\lambda\mu} A^\lambda)(x) (\omega_1 \mathcal{A}_\nu A^\rho{}_{;\rho} + \omega_2 \mathcal{F}_{\rho\nu} A^\rho)(x) > \quad (125)$$

that cancels (124).

Also there is a cross term from (74) and (118):

$$\begin{aligned}
& -i\frac{\omega_2}{\alpha}\kappa^2 \int d^4x \sqrt{-g} \int d^4x' \sqrt{-g'} < (h_{\mu\nu};{}^\nu - \frac{1}{2}h^\nu{}_{\nu;\mu})(x) h_{\lambda'\rho'}(x') > \\
& \mathcal{F}^{\sigma\mu}(x)(g^{\lambda'\gamma'} \mathcal{F}^{\rho'\delta'} - \frac{1}{4}g^{\lambda'\rho'} \mathcal{F}^{\gamma'\delta'})(x') < A_\sigma(x) F_{\gamma'\delta'}(x') > .
\end{aligned} \tag{126}$$

Here the proper time representation again is used, combined with the Ward identity (47) with the term containing the Einstein tensor disregarded, with the result obtained by partial integration:

$$\begin{aligned}
& -i\frac{\omega_2}{\alpha}\kappa^2 \int d^4x \int d^4x' \int_0^\infty \tau d\tau \int_0^1 dt h_{\text{gh},\rho'\mu}(x', x; \frac{1}{\alpha}t\tau) \mathcal{F}^{\sigma\mu}(x) \mathcal{F}^{\rho'\delta'}(x') \\
& (h^\beta{}_{\sigma\delta'}(x, x'; (1-t)\tau);{}^{\gamma'}{}_{;\gamma'} - h^\beta{}_{\sigma}{}^{\gamma'}(x_1, x_2; (1-t)\tau);{}_{\delta';\gamma'})
\end{aligned} \tag{127}$$

with two terms by (79) and (80) and the Ward identity (83):

$$\begin{aligned}
& -\frac{\omega_2}{\alpha}\kappa^2 \int d^4x_1 \int d^4x_2 \int_0^\infty d\tau \frac{\partial}{\partial\tau} \tau \int_0^1 dt h_{\text{gh},\rho'\mu}(x', x; \frac{1}{\alpha}t\tau) \mathcal{F}^{\sigma\mu}(x) \mathcal{F}^{\rho'\delta'}(x') \\
& h^\beta{}_{\sigma\delta'}(x, x'; (1-t)\tau)
\end{aligned} \tag{128}$$

and:

$$\begin{aligned}
& \frac{\omega_2}{\alpha}\kappa^2 \int d^4x \int_0^\infty d\tau h_{\text{gh},\rho\mu}(x, x; \frac{1}{\alpha}\tau) \mathcal{F}_\sigma{}^\mu(x) \mathcal{F}^{\rho\sigma}(x) \\
& -i\frac{\omega_2}{\alpha\beta}\kappa^2 \int d^4x \int d^4x' \int_0^\infty \tau d\tau \int_0^1 dt h_{\text{gh},\rho'\mu}(x', x; \frac{1}{\alpha}t\tau) \mathcal{F}^{\sigma\mu}(x) \mathcal{F}^{\rho'\delta'}(x') \\
& h(x_1, x_2; \frac{1}{\beta}(1-t)\tau);{}_{\sigma;\delta'}.
\end{aligned} \tag{129}$$

(128) is again an expression like (100), containing a proper-time integral of a total differential.

From the ghost action (120) one gets the new one-loop contributions to the effective action:

$$\begin{aligned}
& i\frac{\omega_1}{\sqrt{\alpha\beta}}\kappa^2 \int d^4x \mathcal{A}^\mu(x) \int d^4x' \int_0^\infty \tau d\tau \int_0^1 dt h(x, x'; \frac{t\tau}{\sqrt{\beta}});{}_{\rho;{}^\rho};{}^{\nu'} \\
& (\mathcal{A}^{\lambda'}(x') h_{\text{gh},\lambda'\mu}(x', x; \frac{(1-t)\tau}{\sqrt{\alpha}});{}_{\nu'} + \mathcal{A}_{\nu'};{}^{\lambda'}(x) h_{\text{gh},\lambda'\mu}(x', x; \frac{(1-t)\tau}{\sqrt{\alpha}})) \\
& -\frac{\omega_1}{\sqrt{\alpha}}\kappa^2 \int d^4x \mathcal{A}_\mu(x) (\mathcal{A}^{\lambda'}(x') h_{\text{gh},\lambda'\mu}(x', x; \frac{(1-t)\tau}{\sqrt{\alpha}});{}_{\nu'} \\
& + \mathcal{A}_{\nu'};{}^{\lambda'}(x') h_{\text{gh},\lambda'\mu}(x', x; \frac{(1-t)\tau}{\sqrt{\alpha}}));{}_{\nu'} |_{x' \rightarrow x} \\
& \simeq \frac{\omega_1}{\sqrt{\alpha}}\kappa^2 \int d^4x \mathcal{A}^\mu(x) \int d^4x' \int_0^\infty d\tau \frac{\partial}{\partial\tau} \tau \int_0^1 dt h(x, x'; \frac{t\tau}{\sqrt{\beta}});{}^{\nu'} \\
& (\mathcal{A}^{\lambda'}(x') h_{\text{gh},\lambda'\mu}(x', x; \frac{(1-t)\tau}{\sqrt{\alpha}});{}_{\nu'} + \mathcal{A}_{\nu'};{}^{\lambda'}(x') h_{\text{gh},\lambda'\mu}(x', x; \frac{(1-t)\tau}{\sqrt{\alpha}}))
\end{aligned} \tag{130}$$

and:

$$\begin{aligned}
& -i \frac{\omega_2}{\sqrt{\alpha\beta}} \kappa^2 \int d^4x \mathcal{F}^{\rho\mu}(x) \int d^4x' \mathcal{A}_{\lambda'}(x') \int_0^\infty \tau d\tau \int_0^1 dth(x, x'; \frac{1}{\sqrt{\beta}} t\tau)_{;\rho;\nu';\nu'} \\
& h_{\text{gh}}^{\lambda'\mu}(x', x; \frac{1}{\sqrt{\alpha}}(1-t)\tau) \\
& + \frac{\omega_2}{\sqrt{\alpha}} \kappa^2 \int d^4x \mathcal{F}^{\rho\mu}(x) \mathcal{A}^\lambda(x) h_{\text{gh},\lambda\mu'}(x', x; \tau)_{;\rho} |_{x' \rightarrow x} \\
& \simeq - \frac{\omega_2}{\sqrt{\alpha}} \kappa^2 \int d^4x \mathcal{F}^{\rho\mu}(x) \int d^4x_2 \mathcal{A}_{\lambda'}(x') \int_0^\infty d\tau \frac{\partial}{\partial \tau} \tau \int_0^1 dth(x, x'; \frac{t\tau}{\sqrt{\beta}})_{;\rho} \\
& h_{\text{gh}}^{\lambda'\mu}(x', x; \frac{(1-t)\tau}{\sqrt{\alpha}}) \tag{131}
\end{aligned}$$

by (81), which again are of the same type as (100), with a proper time integral of a total derivative. The final effective action term arising from (120) is:

$$\begin{aligned}
& - \frac{\omega_2}{\sqrt{\alpha}} \kappa^2 \int d^4x \int_0^\infty d\tau h_{\text{gh},\rho\mu}(x, x; \frac{1}{\sqrt{\alpha}}\tau) \mathcal{F}_\sigma{}^\mu(x) \mathcal{F}^{\rho\sigma}(x) \\
& + i \frac{\omega_2}{\sqrt{\alpha\beta}} \kappa^2 \int d^4x \int d^4x' \int_0^\infty \tau d\tau \int_0^1 dth_{\text{gh},\rho'\mu}(x', x; \frac{1}{\sqrt{\alpha}}t\tau) \mathcal{F}^{\sigma\mu}(x) \mathcal{F}^{\rho'\gamma'}(x') \\
& h(x_1, x_2; \frac{1}{\sqrt{\beta}}(1-t)\tau)_{;\sigma;\gamma'}. \tag{132}
\end{aligned}$$

In (129) and (132) one can introduce new variables  $\tau_1 = t\tau$  and  $\tau_2 = (1-t)\tau$ . Then the two expressions cancel formally by rescaling of the variables  $\tau$ ,  $\tau_1$  and  $\tau_2$ . However, this argument is invalidated by quadratic divergences. This is similar to the imperfect cancellation between (53) and (55) and between (86) and (88).

The additional terms in a general gauge are (121), (125), (126), (130), (131) and (132), where the three first terms are modified into the sum of (123), (128) and (129). Formally the sum of these expressions vanishes, but in the proper time representation with the proper time integrals regularized by a lower cut-off the vanishing of the sum is upset by quartic and quadratic divergences. The values of (123), (128), (130) and (131) and the difference between (129) and (132) are all determined in Appendix A.

When using the Ward identity (47) we have disregarded the term on the right-hand side containing the Einstein tensor  $\mathcal{G}^{\mu\nu}$ . When including this term in the calculations reported in this section and also terms of fourth order in  $\kappa$  one generates new terms of the effective action containing one power of the combination  $\mathcal{G}^{\mu\nu} - \mathcal{T}^{\mu\nu}$ , with  $\mathcal{T}^{\mu\nu}$  the background gauge field energy-momentum tensor. The gauge parameter dependence of these terms is removed by the Vilkovisky construction in next-lowest order. The calculation is lengthy, but is important for the use of the Landau-DeWitt gauge condition; an outline is given in Appendix B.

## 5 Momentum space integrals

The flat-space propagators in  $D$  dimensions are:

$$\begin{aligned} < h_{\mu\nu}(x) h_{\lambda'\rho'}(x') >= \int \frac{d^D k}{(2\pi)^D} e^{ik(x-x')} \frac{-i}{k^2 - i\epsilon} \left( g_{\mu\lambda'} g_{\nu\rho'} + g_{\nu\lambda'} g_{\mu\rho'} - \frac{2}{D-2} g_{\mu\nu} g_{\lambda'\rho'} \right. \\ &\quad \left. - (1-\alpha) \frac{1}{k^2} (k_\mu k_{\lambda'} g_{\nu\rho'} + k_\nu k_{\lambda'} g_{\mu\rho'} + k_\mu k_{\rho'} g_{\nu\lambda'} + k_\nu k_{\rho'} g_{\mu\lambda'}) \right) \end{aligned} \quad (133)$$

as well as:

$$< A_\mu(x) A_{\lambda'}(x') >= \int \frac{d^D k}{(2\pi)^D} e^{ik(x-x')} \frac{-i}{k^2 - i\epsilon} (\eta_{\mu\lambda'} - (1-\beta) \frac{1}{k^2} k_\mu k_{\lambda'}) \quad (134)$$

and also:

$$< \xi_\mu(x) \bar{\xi}_{\nu'}(x') >= \sqrt{\alpha} \int \frac{d^D k}{(2\pi)^D} e^{ik(x-x')} \frac{-i}{k^2 - i\epsilon} \eta_{\mu\nu'} \quad (135)$$

with the Ward identity:

$$< (\partial^\mu h_{\mu\nu} - \frac{1}{2} \partial_\nu h^\mu{}_\mu)(x) h_{\lambda'\rho'}(x') >= -\sqrt{\alpha} < (\partial_{\lambda'} \xi_{\rho'} + \partial_{\rho'} \xi_{\lambda'})(x') \xi_\nu(x) >. \quad (136)$$

At  $D = 4$  one gets from (94) when converting it to a momentum space integral:

$$\frac{1}{2} \left( \frac{3}{D'} + (1 - \frac{1}{D'}) \alpha \right) \kappa^2 \int \frac{d^4 k}{(2\pi)^4} \frac{-i}{k^2 - i\epsilon} \int d^4 x \mathcal{F}^{\mu\nu}(x) \mathcal{F}_{\mu\nu}(x). \quad (137)$$

Here was used:

$$\int \frac{d^D k}{(2\pi)^D} f(k^2) k_\mu k_\nu = \frac{1}{D'} \int \frac{d^D k}{(2\pi)^D} f(k^2) k^2 \quad (138)$$

with  $f(k^2)$  arbitrary, where possibly  $D' \neq D$  for quadratic divergences. In [12] and [13] it was argued that the value  $D' = 2$  should be used. Also one gets from (96) for  $D = 4$ :

$$-\frac{3}{4} \kappa^2 (1 - \frac{2}{D'} (1 - \alpha)) \int d^4 x \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu}(x) \int \frac{d^4 k}{(2\pi)^4} \frac{-i}{k^2 - i\epsilon}. \quad (139)$$

The sum of (137) and (139) at general  $D'$  is:

$$(3 - 2\alpha) \left( \frac{1}{D'} - \frac{1}{4} \right) \kappa^2 \int \frac{d^4 k}{(2\pi)^4} \frac{-i}{k^2 - i\epsilon} \int d^4 x \mathcal{F}^{\mu\nu}(x) \mathcal{F}_{\mu\nu}(x) \quad (140)$$

in agreement with Tang and Wu [13].

(140) vanishes at  $D' = 4$ , and we have thus reproduced Pietrykowski's result [4], that the linear divergences of the effective action cancel for all values of  $\alpha$ . Taking instead  $D' = 2$  one gets from (140):

$$\frac{1}{4} (3 + \alpha) \kappa^2 \int \frac{d^4 k}{(2\pi)^4} \frac{-i}{k^2 - i\epsilon} \int d^4 x \mathcal{F}^{\mu\nu}(x) \mathcal{F}_{\mu\nu}(x). \quad (141)$$

in agreement with He, Wang and Xianyu [12].

(140) is considered in connection with (64) converted to a momentum space integral with the replacement  $\mathcal{G}^{\mu\nu} \rightarrow \mathcal{G}^{\mu\nu} - \mathcal{T}^{\mu\nu}$ , where  $\mathcal{T}^{\mu\nu}$  is the background gauge field energy-momentum tensor, and from which one gets at  $D = 4$ :

$$2\alpha\left(\frac{1}{D'} - \frac{1}{4}\right)\kappa^2 \int d^4x \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu}(x) \int \frac{d^D k}{(2\pi)^D} \frac{-i}{k^2 - i\epsilon} \quad (142)$$

where the sum indeed is independent of  $\alpha$ :

$$3\left(\frac{1}{D'} - \frac{1}{4}\right)\kappa^2 \int \frac{d^4 k}{(2\pi)^4} \frac{-i}{k^2 - i\epsilon} \int d^4x \mathcal{F}^{\mu\nu}(x) \mathcal{F}_{\mu\nu}(x). \quad (143)$$

The contributions arising from the Vilkovisky connections (71) and (114) are, keeping in mind that the propagators are transverse:

$$-\left(\frac{1}{D'} - \frac{1}{4}\right)\kappa^2 \int d^D x \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu}(x) \int \frac{d^D k}{(2\pi)^D} \frac{-i}{k^2 - i\epsilon} \quad (144)$$

and:

$$\left(\frac{1}{D'} - \frac{1}{4}\right)\frac{\kappa^2}{4} \int d^D x \mathcal{F}^{\lambda\rho} \mathcal{F}_{\lambda\rho}(x) \int \frac{d^D k}{(2\pi)^D} \frac{-i}{k^2 - i\epsilon} \quad (145)$$

in agreement with Tang and Wu [13]. The sum of (143), (144) and (145) is:

$$\frac{9}{4}\left(\frac{1}{D'} - \frac{1}{4}\right)\kappa^2 \int \frac{d^4 k}{(2\pi)^4} \frac{-i}{k^2 - i\epsilon} \int d^4x \mathcal{F}^{\mu\nu}(x) \mathcal{F}_{\mu\nu}(x). \quad (146)$$

Here a cut-off  $\Lambda$  is introduced in the momentum integral:

$$\int \frac{d^4 k}{(2\pi)^4} \frac{-i}{k^2 - i\epsilon} \simeq \frac{1}{16\pi^2} \Lambda^2 \quad (147)$$

and the sign of the coefficient in (146) indicates at  $D' < 4$  asymptotic freedom. At  $D' = 4$  there is no effect.

In flat space and through use of direct momentum space integration without use of the proper time representation the contributions in general gauges with gauge parameters  $\omega_1, \omega_2$  can be arranged to cancel out. (121) and (125) cancel each other, and here also (126) cancels with (132), converted to a momentum space integral. In both cases the cancellations take place through the Ward identity (136), which must be kept valid in the regularization procedure. The expressions corresponding to (130) and (131) vanish separately.

Momentum space integration seems better off as a regularization procedure compatible with the Vilkovisky construction than the proper-time representation with a lower cut-off in the proper time integral. On the other hand, the proper-time representation allows a direct verification of the removal of the gauge parameter dependence from the finite and logarithmically dependent part of the effective action. Thus a cut-off procedure should be chosen for the proper-time integrals that is modeled after that of momentum space integration.



## 6 Conclusion

The following new results have been obtained in this article: The Vilkovisky construction was reconsidered and criteria for the applicability of a regularization scheme in this context were found. Also, the proper-time representation of the effective action of one-loop quantum gravity was constructed for general gauges, the gauge parameter dependence was investigated, and it was found that the Vilkovisky construction removes from it the finite and logarithmically divergent part but fails to do so from the quadratic and quartic divergences, and these conclusions were extended to the Maxwell-Einstein system. Using momentum-space integration in flat space instead it was found that these defects could be remedied for the Maxwell-Einstein system, suggesting that a modified cut-off procedure of the proper-time integrals should be chosen.

**Acknowledgements:** I am grateful to Professor Francesco Sannino for organizing a meeting, which triggered this investigation, on the occasion of my retirement, and to Professor Bo-Sture Skagerstam for a very inspiring correspondence.

## A Heat kernel expansion

The ghost heat kernel defined by (43) and (44) has the expansion [11]:

$$h_{\text{gh}\mu\xi'}(x, x'; \tau) = \frac{-i}{16\pi^2} \frac{1}{\tau^2} e^{i\frac{\sigma}{2\tau}} \Delta^{\frac{1}{2}} \sum_{n=0}^{\infty} a_{n,\text{gh}\mu\xi'}(x, x') (i\tau)^n \quad (148)$$

where  $\sigma$  is the geodesic interval between  $x$  and  $x'$  and  $\Delta$  is the so-called Van Vleck determinant. At coinciding points one has:

$$a_{0,\text{gh},\mu\nu'}(x, x') \big|_{x' \rightarrow x} = g_{\mu\nu}, \quad a_{1,\text{gh},\mu\nu'}(x, x') \big|_{x' \rightarrow x} \simeq -\frac{R}{6} g_{\mu\nu} - R_{\mu\nu}. \quad (149)$$

For the scalar heat kernel  $h(x, x'; \tau)$  defined by (81) a corresponding expansion applies, with:

$$a_0(x, x') \big|_{x' \rightarrow x} = 1, \quad a_1(x, x') \big|_{x' \rightarrow x} = -\frac{R}{6}. \quad (150)$$

Also:

$$\sigma_{;\lambda;\rho'} \simeq -g_{\lambda\rho'} \quad (151)$$

for  $x' \simeq x$ . Hence it follows from (148):

$$h_{\text{gh}\mu\xi'}(x, x'; \tau)_{;\lambda;\rho'} = -g_{\lambda\rho'} \frac{1}{32\pi^2 \tau^3} e^{i\frac{\sigma}{2\tau}} \Delta^{\frac{1}{2}} \sum_{n=0}^{\infty} a_{n,\text{gh}\mu\xi'}(x, x') (i\tau)^n + \dots \quad (152)$$

where the remaining terms vanish at coinciding points. The graviton heat kernel  $h_{\mu\nu,\xi'\eta'}^\alpha(x, x'; \tau)$  is by (51) and (149):

$$h_{\mu\nu,\xi'\eta'}^\alpha(x, x'; \tau) = \frac{-i}{16\pi^2} \frac{1}{\tau^2} e^{i\frac{\sigma}{2\tau}} \Delta^{\frac{1}{2}} (\alpha(g_{\mu\xi'}g_{\nu\eta'} + g_{\mu\eta'}g_{\nu\xi'} - g_{\mu\nu}g_{\xi'\eta'}) + \dots) \quad (153)$$

The leading divergence of (94) at  $\alpha = 1$  is determined by the quantity:

$$\begin{aligned} & \int d^4x' \int_0^1 dt h_{\mu\nu,\xi'\eta'}^\alpha(x, x'; t\tau) \big|_{\alpha=1} (g^{\eta'\gamma'} \mathcal{F}^{\xi'\delta'} - \frac{1}{4} g^{\xi'\eta'} \mathcal{F}^{\gamma'\delta'}) (x') \\ & (h_{\rho\delta'}^\beta(x, x'; (1-t)\tau)_{;\lambda;\gamma'} - h_{\rho\gamma'}^\beta(x, x'; (1-t)\tau)_{;\lambda;\delta'} - h_{\lambda\beta'}^\beta(x, x'; (1-t)\tau)_{;\rho;\alpha'} \\ & + h_{\lambda\gamma'}^\beta(x, x'; (1-t)\tau)_{;\rho;\delta'}) \\ & \simeq -\frac{1}{16\pi^2\tau^3} \sqrt{-g} (g_{\mu\xi}g_{\nu\eta} + g_{\mu\eta}g_{\nu\xi} - g_{\mu\nu}g_{\xi\eta}) (g^{\eta\gamma} \mathcal{F}^{\xi\delta} - \frac{1}{4} g^{\xi\eta} \mathcal{F}^{\gamma\delta}) (x) \\ & (g_{\rho\delta}g_{\lambda\gamma} - g_{\lambda\delta}g_{\rho\gamma}) \end{aligned} \quad (154)$$

where the evaluation for simplicity can be carried out in flat space by Fourier transformation. In (104) one encounters:

$$\begin{aligned} & \int d^4x \int_0^1 dt h_{\text{gh}\mu\xi'}(x, x'; t\frac{1}{\alpha}\tau) \mathcal{F}^{\mu\rho}(x) \\ & (h_{\rho\delta'}^\beta(x, x'; (1-t)\tau)_{;\gamma';\eta'} - h_{\rho\gamma'}^\beta(x, x'; (1-t)\tau)_{;\delta';\eta'}) \\ & \simeq \frac{\alpha^2}{4} \frac{1}{16\pi^2\tau^3} \sqrt{-g} (g_{\gamma'\eta'} \mathcal{F}^{\xi'\delta'} - g_{\delta'\eta'} \mathcal{F}^{\xi'\gamma'}) (x') \end{aligned} \quad (155)$$

where the evaluation again most simply is carried out in flat space.

The effective action in a general gauge in the heat-kernel representation also contains the nonvanishing expressions (123), (128), (130) and (131). They all contain a total derivative in the proper time integral and vanish in a formal sense in the same way as (100). Nevertheless, they contain quartic or quadratic divergences. Also the effective action contains (129) and (132), which cancel formally, but in fact have a quadratically divergent sum depending on the gauge parameters. The evaluation of these quantities is sketched below; the calculation is most simply carried out in flat space and leads to the following intermediary results:

$$\begin{aligned} & \int d^4x' \int_0^1 dt h_{\text{gh},\eta\mu'}(x, x'; \frac{1}{\alpha}t\tau) \mathcal{F}^{\mu'\lambda'}(x') h_{\lambda'\beta}^\beta(x', x; (1-t)\tau) \\ & \simeq -\frac{1}{4} \alpha(3 + \beta) \frac{i}{16\pi^2\tau^2} \mathcal{F}_{\eta\beta}(x) \end{aligned} \quad (156)$$

and also:

$$\begin{aligned} & \int d^4x'' \int_0^1 dt h_{\text{gh},\mu\lambda''}(x, x''; \frac{1}{\alpha}t\tau) \mathcal{A}^{\lambda''}(x'') h(x'', x'; \frac{1}{\beta}(1-t)\tau) \\ & \simeq -\alpha\beta \frac{i}{16\pi^2\tau^2} \mathcal{A}_\mu(x) \end{aligned} \quad (157)$$

that implies:

$$\begin{aligned} & \int d^4 x'' \int_0^1 dt h_{\text{gh}, \mu \lambda''}(x, x''; \frac{1}{\alpha} t \tau) \mathcal{A}^{\lambda''}(x'') h(x'', x'; \frac{1}{\beta} (1-t) \tau)_{, \nu'} \\ & \simeq -\frac{1}{2} \alpha \beta \frac{i}{16 \pi^2 \tau^2} \mathcal{A}_{\mu, \nu'}(x). \end{aligned} \quad (158)$$

(123) is by the Ward identity (83) the sum of three terms:

$$\begin{aligned} & -\frac{1}{2} \frac{\omega_2^2}{\alpha} \kappa^2 \int d^4 x \int d^4 x' \int_0^\infty d\tau \frac{\partial}{\partial \tau} \tau \int_0^1 dt h_{\text{gh}, \lambda' \mu}(x', x; \frac{1}{\alpha} t \tau) \\ & \mathcal{F}^{\sigma \mu}(x) \mathcal{F}^{\gamma' \lambda'}(x') h^\beta_{\sigma \gamma'}(x, x'; (1-t) \tau) \\ & \simeq -\frac{1}{8} (3 + \beta) \omega_2^2 \frac{i}{16 \pi^2 \tau} \Big|_{\tau \simeq 0} \kappa^2 \int d^4 x \sqrt{-g} \mathcal{F}^{\mu \nu} \mathcal{F}_{\mu \nu}(x) \end{aligned} \quad (159)$$

by (156), and also:

$$\begin{aligned} & \frac{\omega_1 \omega_2}{\alpha} \kappa^2 \int d^4 x \int d^4 x' \int_0^\infty d\tau \frac{\partial}{\partial \tau} \tau \int_0^1 dt h_{\text{gh}, \lambda' \mu}(x', x; \frac{1}{\alpha} t \tau) \\ & \mathcal{A}^\mu(x) \mathcal{F}^{\gamma' \lambda'}(x') h(x, x'; \frac{1}{\beta} (1-t) \tau)_{, \gamma'} \\ & \simeq \frac{1}{2} \beta \omega_1 \omega_2 \frac{i}{16 \pi^2 \tau} \Big|_{\tau \simeq 0} \kappa^2 \int d^4 x \sqrt{-g} \mathcal{F}^{\mu \nu} \mathcal{A}_{\nu, \mu}(x) \end{aligned} \quad (160)$$

by (158), and finally:

$$\begin{aligned} & \frac{1}{2} \frac{\omega_1^2}{\alpha} \kappa^2 \int d^4 x \int d^4 x' \int_0^\infty d\tau \frac{\partial}{\partial \tau} \tau \int_0^1 dt h_{\text{gh}, \lambda' \mu}(x', x; \frac{1}{\alpha} t \tau) \\ & \mathcal{A}^\mu(x) \mathcal{A}^{\lambda'}(x') h(x, x'; \frac{1}{\beta} (1-t) \tau)_{, \sigma; \sigma} \\ & = -i \frac{1}{2} \frac{\beta \omega_1^2}{\alpha} \kappa^2 \int d^4 x \int d^4 x' \int_0^\infty d\tau \frac{\partial^2}{\partial \tau^2} \tau \int_0^1 dt h_{\text{gh}, \mu_2 \mu_1}(x_2, x_1; \frac{1}{\alpha} t \tau) \\ & \mathcal{A}^\mu(x) \mathcal{A}^{\lambda'}(x') h(x, x'; \frac{1}{\beta} (1-t) \tau) \\ & + i \frac{1}{2} \frac{\beta \omega_1^2}{\alpha} \kappa^2 \int d^4 x \int_0^\infty d\tau \frac{\partial}{\partial \tau} h_{\text{gh}, \mu \nu}(x, x; \frac{1}{\alpha} \tau) \mathcal{A}^\mu(x) \mathcal{A}^\nu(x) \\ & \simeq -\frac{1}{2} \beta (\alpha + \beta) \omega_1^2 \frac{1}{16 \pi^2 \tau^2} \Big|_{\tau \simeq 0} \frac{1}{\tau^2} \kappa^2 \int d^4 x \sqrt{-g} \mathcal{A}^\mu \mathcal{A}_\mu(x) \end{aligned} \quad (161)$$

by (157), where only the quartic divergence was kept.

The value of (128) is by (156):

$$\frac{1}{4} (3 + \beta) \omega_2 \frac{i}{16 \pi^2 \tau} \Big|_{\tau \simeq 0} \kappa^2 \int d^4 x \sqrt{-g} \mathcal{F}^{\mu \nu} \mathcal{F}_{\mu \nu}(x). \quad (162)$$

(130) contains two terms:

$$\frac{\omega_1}{\sqrt{\alpha}} \kappa^2 \int d^4 x \mathcal{A}^\mu(x) \int d^4 x' \int_0^\infty d\tau \frac{\partial}{\partial \tau} \tau \int_0^1 dt h(x, x'; \frac{t \tau}{\sqrt{\beta}})_{, \nu'}$$

$$\begin{aligned}
& \mathcal{F}^{\lambda'}{}_{\nu'}(x') h_{\text{gh}, \lambda' \mu}(x', x; \frac{(1-t)\tau}{\sqrt{\alpha}})) \\
& \simeq -\frac{1}{2} \sqrt{\beta} \omega_1 \frac{i}{16\pi^2} \frac{1}{\tau} \Big|_{\tau \simeq 0} \kappa^2 \int d^4x \sqrt{-g} \mathcal{A}_{\nu, \mu} \mathcal{F}^{\mu\nu}
\end{aligned} \tag{163}$$

by (158), and:

$$\begin{aligned}
& -\frac{\omega_1}{\sqrt{\alpha}} \kappa^2 \int d^4x_1 \mathcal{A}^\mu(x) \int d^4x' \int_0^\infty d\tau \frac{\partial}{\partial \tau} \tau \int_0^1 dt \\
& h(x, x'; \frac{t\tau}{\sqrt{\beta}})_{;\nu'}{}^{\nu'} \mathcal{A}^{\lambda'}(x') h_{\text{gh}, \lambda' \mu}(x', x; \frac{(1-t)\tau}{\sqrt{\alpha}}) \\
& \simeq \sqrt{\beta(\alpha + \beta)} \omega_1 \frac{1}{16\pi^2} \frac{1}{\tau^2} \Big|_{\tau \simeq 0} \kappa^2 \int d^4x \sqrt{-g} \mathcal{A}^\mu \mathcal{A}_\mu
\end{aligned} \tag{164}$$

cp. (161), where only the quartic divergence was determined.

(131) is by (158):

$$-\frac{1}{2} \sqrt{\beta} \omega_2 \frac{i}{16\pi^2} \frac{1}{\tau} \Big|_{\tau \simeq 0} \kappa^2 \int d^4x \sqrt{-g} \mathcal{A}_{\nu, \mu} \mathcal{F}^{\mu\nu}. \tag{165}$$

We then consider (129) and (132), using flat space heat kernels. The first term of (129) has the quadratic divergence:

$$\alpha \omega_2 \frac{i}{16\pi^2} \frac{1}{\tau} \Big|_{\tau \simeq 0} \kappa^2 \int d^4x \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu}(x) \tag{166}$$

while the second term of (129) is evaluated by means of (157) which implies:

$$\begin{aligned}
& \int d^4x'' \int_0^1 dt h_{\text{gh}, \mu \lambda''}(x, x''; \frac{1}{\alpha} t\tau) \mathcal{F}^{\lambda'' \rho''}(x'') h(x'', x'; \frac{1}{\beta} (1-t)\tau)_{;\rho''; \sigma'} \\
& \simeq -\frac{1}{4} \alpha \beta (\alpha + \beta) \frac{1}{16\pi^2 \tau^3} \mathcal{F}_{\mu \sigma'}(x)
\end{aligned} \tag{167}$$

and so the second term of (129) is:

$$-\frac{1}{4} (\alpha + \beta) \omega_2 \frac{i}{16\pi^2} \frac{1}{\tau} \Big|_{\tau \simeq 0} \kappa^2 \int d^4x \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu}(x). \tag{168}$$

The value of (132) is obtained from (166) and (168) by changing sign and replacing  $\alpha$  by  $\sqrt{\alpha}$  and  $\beta$  by  $\sqrt{\beta}$ .

## B The Vilkovisky construction in next-lowest order

In the Landau-De Witt gauge we require that the gauge condition (23) is chosen such that only the Christoffel connection coupling term (24) survives in (22). For the Maxwell-Einstein system the form of this gauge condition is:

$$\chi = \chi_\lambda = 0 \tag{169}$$

with:

$$\chi = -\sqrt{-g}A_{\mu;{}^\mu} \quad (170)$$

and:

$$\chi_\lambda = -\sqrt{-g}(h_{\lambda\mu;{}^\mu} - \frac{1}{2}h_{\mu;{}^\mu;\lambda} + \kappa(A_{\mu;{}^\mu}\mathcal{A}_\lambda + \mathcal{F}_{\mu\lambda}A^\mu)) \quad (171)$$

which is the gauge condition (117) with  $\omega_1 = \omega_2 = 1$  (the sign is unimportant). However, it was shown that the Vilkovisky construction of quantum gravity is sufficient to remove the dependence of the effective action on the dependence on the gauge parameter  $\alpha$ , and also that the additional terms of the effective action in gauges with general values of the gauge parameters  $\omega_1$  and  $\omega_2$  formally cancel. One has to conclude that the Landau-DeWitt gauge appropriate for the Maxwell-Einstein system effective action taken to second order in  $\kappa\mathcal{F}_{\mu\nu}$  is the same as that of quantum gravity, which is obtained by taking  $\kappa \rightarrow 0$  in (171), while the Landau-DeWitt gauge condition obtained from (171) itself only is relevant to fourth order in  $\kappa\mathcal{F}_{\mu\nu}$ . That this is indeed the case follows from a detailed examination of the terms of (22) in this order of Maxwell-Einstein theory.

The Ward identity corresponding to (15) is first determined. The graviton field two-point function  $\langle h_{\mu\nu}(x)h_{\xi'\eta'}(x') \rangle$  is related to the heat kernel  $h_{\mu\nu,\xi'\eta'}^\alpha(x, x'; \tau)$  through (65) and the ghost two-point function  $\langle \xi_\mu(x)\bar{\xi}_\nu(y) \rangle$  is expressed through the heat kernel  $h_{\text{gh},\mu,\xi'}(x, x'; \tau)$  in (70). (47) implies by (65) and (70) the Ward identity (15) specialized to quantum gravity and relating graviton and ghost two-point functions, cp. (136):

$$\begin{aligned} & \langle (h_{\mu\nu} - \frac{1}{2}h_{\mu;{}^\mu;\nu})(x)h_{\lambda\rho}(y) \rangle \simeq -\sqrt{\alpha} \langle (\xi_{\lambda;\rho} + \xi_{\rho;\lambda})(y)\bar{\xi}_\nu(x) \rangle \\ & + i\sqrt{\alpha} \int d^4w \sqrt{-g} \langle \xi^\sigma(w)\bar{\xi}_\nu(x) \rangle \mathcal{G}^{\omega\delta}(w) \langle (2h_{\omega\sigma;\delta} - h_{\omega\delta;\sigma})(w)h_{\lambda\rho}(y) \rangle . \end{aligned} \quad (172)$$

In the same way one gets from (49), using also (47) with the term involving the Einstein tensor disregarded:

$$\begin{aligned} & \alpha \frac{\partial}{\partial \alpha} \langle h_{\mu\nu}(x)h_{\lambda'\rho'}(x') \rangle \\ & \simeq i \int d^4x'' \sqrt{-g''} \langle (\xi_{\mu;\nu} + \xi_{\nu;\mu})(x)\bar{\xi}^{\sigma''}(x'') \rangle \langle (\xi_{\lambda';\rho'} + \xi_{\rho';\lambda'})(x')\bar{\xi}_{\sigma''}(x'') \rangle . \end{aligned} \quad (173)$$

If the Ward identity (172) is used with the second term on the right hand side included, and the replacement  $\mathcal{G}^{\mu\nu} \rightarrow \mathcal{G}^{\mu\nu} - \mathcal{T}^{\mu\nu}$  next is made, additional terms arise from (121) and (126):

$$\begin{aligned} & \frac{1}{\alpha} \kappa^2 \int d^4x_1 \sqrt{-g_1} g^{\mu_1\nu_1} \int d^4x_2 \sqrt{-g} g^{\mu_2\nu_2} \int d^4w \sqrt{-g} \langle \xi^\sigma(w)\bar{\xi}_{\mu_1}(x_1) \rangle \\ & (\mathcal{G}^{\omega\delta} - \mathcal{T}^{\omega\delta})(w) \langle (\xi_{\sigma;\omega;\delta} - R^v_{\omega\delta\sigma}\xi_v)(w)\bar{\xi}_{\mu_2}(x_2) \rangle \\ & \langle (\omega_1\mathcal{A}_{\nu_1}A^{\rho_1}_{\phantom{\rho_1};\rho_1} + \omega_2\mathcal{F}_{\rho_1\nu_1}A^{\rho_1})(x_1)(\omega_1\mathcal{A}_{\nu_2}A^{\rho_2}_{\phantom{\rho_2};\rho_2} + \omega_2\mathcal{F}_{\rho_2\nu_2}A^{\rho_2})(x_2) \rangle \end{aligned} \quad (174)$$

and:

$$\begin{aligned}
& \frac{1}{\sqrt{\alpha}} \kappa^2 \omega_2 \int d^4 x_1 \sqrt{-g_1} \int d^4 x_2 \sqrt{-g_2} \int d^4 w \sqrt{-g} < \xi^\sigma(w) \bar{\xi}_{\mu_1}(x) > \\
& (\mathcal{G}^{\omega\delta} - \mathcal{T}^{\omega\delta})(w) < (2h_{\omega\sigma;\delta} - h_{\omega\delta;\sigma})(w) h_{\mu_2\nu_2}(x_2) > \\
& g^{\mu_1\rho_1} \mathcal{F}_{\lambda_1\rho_1}(x_1) (g^{\mu_2\lambda_2} \mathcal{F}^{\nu_2\rho_2} - \frac{1}{4} g^{\mu_2\nu_2} \mathcal{F}^{\lambda_2\rho_2})(x_2) < A^{\lambda_1}(x_1) F_{\lambda_2\rho_2}(x_2) > .
\end{aligned} \tag{175}$$

These expressions are partially of fourth order in  $\kappa$  and in the background field  $\mathcal{A}_\mu$ . The presence of these fourth order terms can be proven directly by a lengthy calculation.

$N^{\alpha\beta}$  now has components at first order in  $\kappa$ :

$$\begin{aligned}
& N^{\xi_\alpha(x)c(y)} \\
& \simeq \kappa \frac{1}{\sqrt{\alpha\beta}} \int d^4 w \sqrt{-g} < \xi_\alpha(x) ((\bar{\xi}_\lambda \mathcal{A}^\lambda);^\mu + \bar{\xi}_\lambda(w) \mathcal{F}^{\lambda\mu})(w) > < c_{,\mu}(w) \bar{c}(y) > .
\end{aligned} \tag{176}$$

At second order (63) is modified to:

$$\begin{aligned}
& N^{\xi_\mu(x)\xi_\nu(y)} \simeq i \frac{1}{\sqrt{\alpha}} < \xi_\mu(x) \bar{\xi}_\nu(y) > + \kappa^2 \frac{1}{\alpha} \int d^4 w \int d^4 z \sqrt{-g} < \xi_\mu(x) \bar{\xi}_\omega(w) > \mathcal{F}^{\omega\lambda}(w) \\
& \Pi_{\lambda\rho}(w, z) \mathcal{F}^{\sigma\rho}(z) < \xi_\sigma(z) \bar{\xi}_\nu(y) >
\end{aligned} \tag{177}$$

where  $\Pi_{\lambda\rho}$  was introduced in (110).

The projection operator (69) is unmodified in lowest order but gets an additional term at second order in  $\kappa$ :

$$\begin{aligned}
& \Delta \Pi^{h_{\mu\nu}(x)}_{h_{\lambda\rho}(y)} \\
& = -\frac{1}{\alpha} \kappa^{24} \sqrt{-g} \int d^4 w \int d^4 z < (\xi_{\mu;\nu} + \xi_{\nu;\mu})(x) \bar{\xi}^\gamma(w) > {}^4\sqrt{-g} \mathcal{F}_{\gamma\alpha}(w) \\
& \Pi^{\alpha\beta}(w, z) \mathcal{F}_{\eta\beta}(z) {}^4\sqrt{-g} < \xi^\eta(z) \frac{1}{2} (\bar{\xi}^\lambda;^\rho + \bar{\xi}^\rho;^\lambda - g^{\lambda\rho} \bar{\xi}_\sigma;^\sigma)(y) > {}^4\sqrt{-g}.
\end{aligned} \tag{178}$$

A mixed projection operator is by (73) and (176):

$$\begin{aligned}
& \Pi^{A_\mu(x)}_{h_{\lambda\rho}(y)} \\
& = -i \frac{1}{\sqrt{\alpha}} \kappa \int d^4 w \Pi_\mu{}^\nu(x, w) {}^4\sqrt{-g} \mathcal{F}_{\sigma\nu}(w) \frac{1}{\sqrt{\alpha}} < \xi^\sigma(w) \frac{1}{2} (\bar{\xi}^\lambda;^\rho + \bar{\xi}^\rho;^\lambda - g^{\lambda\rho} \bar{\xi}_\omega;^\omega)(y) > {}^4\sqrt{-g}.
\end{aligned} \tag{179}$$

In the Vilkovisky construction new terms of order  $\kappa^2$  in (22) originate from:

$$\frac{1}{2} \int d^4 x \int d^4 y \int d^4 z \int d^4 w \int d^4 u \int d^4 t S_{h_{\mu\nu}(x)} R^{h_{\mu\nu}(x)}_{\xi_\omega(y), h_{\lambda\rho}(z)} N^{\xi_\omega(y)\xi_\sigma(w)}$$

$$\begin{aligned}
& R^{h_{\xi\eta}(u)}_{\xi\sigma(w)} G_{h_{\xi\eta}(u)h_{\alpha\beta}(t)} < h_{\alpha\beta}(t)h_{\lambda\rho}(z) > \\
& + \frac{1}{2} \int d^4x \int d^4y \int d^4z \int d^4w \int d^4u \int d^4t S_{,h_{\mu\nu}(x)} R^{h_{\mu\nu}(x)}_{\xi\omega(y),h_{\lambda\rho}(z)} \\
& (N^{\xi\omega(y)\xi\sigma(w)} R^{A_\xi(u)}_{\xi\sigma(w)} + N^{\xi\omega(y)c(w)} R^{A_\xi(u)}_{c(w)}) G_{A_\xi(u)A_\alpha(t)} < A_\alpha(t)h_{\lambda\rho}(z) >^4 \sqrt{-g} \\
& (180)
\end{aligned}$$

and:

$$\begin{aligned}
& \frac{1}{2} \int d^4x \int d^4y \int d^4z \int d^4w \int d^4u \int d^4t \int d^4r S_{,h_{\mu\nu}(x)} R^{h_{\mu\nu}(x)}_{\xi\omega(y),h_{\lambda\rho}(z)} N^{\xi\omega(y)\xi\sigma(w)} \\
& R^{h_{\xi\eta}(u)}_{\xi\sigma(w)} G_{h_{\xi\eta}(u)h_{\alpha\beta}(t)} < h_{\alpha\beta}(t)h_{\gamma\delta}(r) >^4 \sqrt{-g} \Pi^{h_{\lambda\rho}(z)}_{h_{\gamma\delta}(r)} \\
& + \frac{1}{2} \int d^4x \int d^4y \int d^4z \int d^4w \int d^4u \int d^4t \int d^4r S_{,h_{\mu\nu}(x)} R^{h_{\mu\nu}(x)}_{\xi\omega(y),h_{\lambda\rho}(z)} N^{\xi\omega(y)\xi\sigma(w)} \\
& R^{h_{\xi\eta}(u)}_{\xi\sigma(w)} G_{h_{\xi\eta}(u)h_{\alpha\beta}(t)} < h_{\alpha\beta}(t)A_\gamma(r) >^4 \sqrt{-g} \Pi^{h_{\lambda\rho}(z)}_{A_\gamma(r)} \\
& + \frac{1}{2} \int d^4x \int d^4y \int d^4z \int d^4w \int d^4u \int d^4t \int d^4r S_{,h_{\mu\nu}(x)} R^{h_{\mu\nu}(x)}_{\xi\omega(y),h_{\lambda\rho}(z)} \\
& (N^{\xi\omega(y)\xi\sigma(w)} R^{A_\xi(u)}_{\xi\sigma(w)} + N^{\xi\omega(y)c(w)} R^{A_\xi(u)}_{c(w)}) G_{A_\xi(u)A_\alpha(t)} < A_\alpha(t)h_{\gamma\delta}(r) >^4 \sqrt{-g} \Pi^{h_{\lambda\rho}(z)}_{h_{\gamma\delta}(r)} \\
& + \frac{1}{2} \int d^4x \int d^4y \int d^4z \int d^4w \int d^4u \int d^4t \int d^4r S_{,h_{\mu\nu}(x)} R^{h_{\mu\nu}(x)}_{\xi\omega(y),g_{\lambda\rho}(z)} \\
& (N^{\xi\omega(y)\xi\sigma(w)} R^{A_\xi(u)}_{\xi\sigma(w)} + N^{\xi\omega(y)c(w)} R^{A_\xi(u)}_{c(w)}) G_{A_\xi(u)A_\alpha(t)} < A_\alpha(t)A_\gamma(r) >^4 \sqrt{-g} \Pi^{h_{\lambda\rho}(z)}_{A_\gamma(r)}. \\
& (181)
\end{aligned}$$

These expressions are now analyzed and shown to cancel with (174) and (175); also it is shown that they vanish, if the Landau-DeWitt gauge conditions (169) are imposed.

The parts of (180) and (181) involving the graviton correlation function  $< h_{\alpha\beta}(t)h_{\gamma\delta}(r) >$  at second order in  $\kappa$  constructed by means of the couplings (74) and (76) involves two factors  $S_{,h_{\mu\nu}}$ ; the argument amounts to using (172) with the replacement  $\mathcal{G}^{\mu\nu} \rightarrow \mathcal{G}^{\mu\nu} - \mathcal{T}^{\mu\nu}$  in the second term on the right hand side. Consequently these terms are disregarded in the approximation where only one derivative of the classical action is kept in (22).

There is an extra term in (180) from the second order term of (177):

$$\begin{aligned}
& \frac{1}{2} \frac{1}{\alpha} \kappa^2 \int d^4x \sqrt{-g} \int d^4y \int d^4w \int d^4z \sqrt{-g} \sqrt{-g} (\mathcal{G}^{\mu\nu} - \mathcal{T}^{\mu\nu})(x) \\
& < (h_{\omega\tau;\tau} - \frac{1}{2} h_{\tau\tau;\omega})(y) (h_{\mu\lambda}\xi^\lambda_{;\nu} + h_{\nu\lambda}\xi^\lambda_{;\mu} + \xi^\lambda h_{\mu\nu;\lambda})(x) \bar{\xi}_\sigma(w) > \mathcal{F}^{\sigma\lambda}(w) \\
& \Pi_{\lambda\rho}(w, z) \mathcal{F}^{\sigma\rho}(z) < \xi_\sigma(z) \bar{\xi}^\omega(y) > \\
& (182)
\end{aligned}$$

that vanishes in the Landau-DeWitt gauge to order  $\kappa^2$  since the additional term in (171) makes the expression  $O(\kappa^3)$ . The corresponding term of (181) vanishes.

In (180) and (181) the following combination is present:

$$\begin{aligned}
& \int d^4w \int d^4u (N^{\xi\omega(y)\xi\sigma(w)} R^{A_\xi(u)}_{\xi\sigma(w)} + N^{\xi\omega(y)c(w)} R^{A_\xi(u)}_{c(w)}) G_{A_\xi(u)A_\alpha(t)} A_\alpha(t) \\
& = -i \frac{1}{\sqrt{\alpha}} \kappa \int d^4w A^\lambda(x) \Pi_{\lambda\mu}(x, w) \mathcal{F}^{\mu\rho}(w) < \xi_\rho(w) \bar{\xi}_\omega(y) > \\
& (183)
\end{aligned}$$

and thus one gets by (179):

$$\begin{aligned}
& \frac{1}{2} \int d^4x \int d^4y \int d^4z \int d^4w \int d^4u \int d^4t \int d^4r S_{,h\mu\nu(x)} R^{h\mu\nu(x)}_{\xi\omega(y),g\lambda\rho(z)} \\
& (N^{\xi\omega(y)\xi\sigma(w)} R^{A\xi(u)}_{\xi\sigma(w)} + N^{\xi\omega(y)c(w)} R^{A\xi(u)}_{c(w)}) G_{A\xi(u)A\alpha(t)} < A_\alpha(t) A_\gamma(r) >^4 \sqrt{-g} \Pi^{h\lambda\rho(z)}_{A_\gamma(r)} \\
& \simeq \frac{1}{\alpha} \kappa^2 \int d^4x \sqrt{-g} \int d^4y \int d^4w \int d^4t (\mathcal{G}^{\mu\nu}(x) - \mathcal{T}^{\mu\nu})(x) \\
& \mathcal{F}^{\omega\rho}(w)^4 \sqrt{-g} \Pi_{\omega\lambda}(w, t)^4 \sqrt{-g} < A^\lambda(t) A_\gamma(r) > \mathcal{F}^{\sigma\gamma}(r) \\
& (< \xi_{v;\mu}(x) \bar{\xi}_\sigma(w) > < \xi^v_{;\nu}(x) \bar{\xi}_\rho(y) > - R_{v\mu\phi\nu}(x) < \xi^v(x) \bar{\xi}_\sigma(w) > < \xi^\phi(x) \bar{\xi}_\rho(y) >). \quad (184)
\end{aligned}$$

In (181) one gets by (178):

$$\begin{aligned}
& \int d^4w \int d^4u \int d^4t R^{g\xi\eta(w)}_{\xi_\mu(x)} G_{\xi\eta,\alpha\beta}(w, u)^4 \sqrt{-g} < h_{\alpha\beta}(u) h_{\sigma\omega}(t) >^4 \sqrt{-g} \Pi^{g\lambda\rho(z)}_{g\sigma\omega(t)} \\
& \rightarrow i\kappa^{24} \sqrt{-g} \int d^4w \int d^4z \sqrt{-g} \int d^4u \sqrt{-g} \mathcal{F}^{\omega\sigma}(w) < \xi_\omega(w) \bar{\xi}_\mu(x) > \Pi_{\sigma\gamma}(w, u)^4 \sqrt{-g} \mathcal{F}^{\beta\gamma}(u) \\
& < \xi_\beta(u) (\bar{\xi}_{\lambda;\rho} + \bar{\xi}_{\rho;\lambda})(z) >. \quad (185)
\end{aligned}$$

Also one finds in (181) by (172):

$$\begin{aligned}
& \int d^4w \int d^4u \int d^4t R^{g\xi\eta(w)}_{\xi_\mu(x)} G_{\xi\eta,\alpha\beta}(w, u)^4 \sqrt{-g} < h_{\alpha\beta}(u) A_\gamma(t) >^4 \sqrt{-g} \Pi^{g\lambda\rho(z)}_{A_\gamma(t)} \\
& \rightarrow -i\kappa^{24} \sqrt{-g} \int d^4w^4 \sqrt{-g} \int d^4u^4 \sqrt{-g} \mathcal{F}^{\omega\sigma}(w) < \xi_\omega(w) \bar{\xi}_\mu(x) > \Pi_{\sigma\gamma}(w, u)^4 \sqrt{-g} \mathcal{F}^{\beta\gamma}(u) \\
& < \xi_\beta(u) (\bar{\xi}_{\lambda;\rho} + \bar{\xi}_{\rho;\lambda})(z) > \quad (186)
\end{aligned}$$

where the correlation function  $< A_\mu(x) h_{\lambda\rho}(z) >$  was formed by means of the coupling (74) with the splitting (92), and (186) cancels with (185).

Other higher order terms constructed only by means of the coupling (74) are next considered. Using (183) one gets by (33):

$$\begin{aligned}
& \int d^4y \int d^4z \int d^4w \int d^4u \int d^4t R^{h\mu\nu(x)}_{\xi\omega(y),h\lambda\rho(z)} \\
& (N^{\xi\omega(y)\xi\sigma(w)} R^{A\xi(u)}_{\xi\sigma(w)} + N^{\xi\omega(y)c(w)} R^{A\xi(u)}_{c(w)}) G_{A\xi(u)A\alpha(t)} < A_\alpha(t) h_{\lambda\rho}(z) > \\
& \rightarrow \frac{1}{\sqrt{\alpha}} \kappa^3 \int d^4y \int d^4w \int d^4u \mathcal{F}_{\sigma\omega}(y)^4 \sqrt{-g} \Pi^{\sigma\tau}(y, w)^4 \sqrt{-g} < A_\tau(w) F_{\xi\eta}(u) > \\
& (g^{\alpha\xi} \mathcal{F}^{\beta\eta} - \frac{1}{4} g^{\alpha\beta} \mathcal{F}^{\xi\eta})(u) < h_{\alpha\beta}(u) (h_{\mu\lambda} \xi^\lambda_{;\nu} + h_{\nu\lambda} \xi^\lambda_{;\mu} + \xi^\lambda h_{\mu\nu;\lambda})(x) \bar{\xi}^\omega(y) >. \quad (187)
\end{aligned}$$

(187) contributes to the effective action through (180):

$$\begin{aligned}
& -\frac{1}{\sqrt{\alpha}} \kappa^2 \int d^4x \sqrt{-g} \int d^4y \int d^4w \int d^4u \sqrt{-g} (\mathcal{G}^{\mu\nu} - \mathcal{T}^{\mu\nu})(x) \\
& \mathcal{F}_{\sigma\omega}(y)^4 \sqrt{-g} \Pi^{\sigma\tau}(y, w)^4 \sqrt{-g} < A_\tau(w) F_{\xi\eta}(u) > \\
& (g^{\alpha\xi} \mathcal{F}^{\beta\eta} - \frac{1}{4} g^{\alpha\beta} \mathcal{F}^{\xi\eta})(u) < h_{\alpha\beta}(u) (h_{\mu\lambda} \xi^\lambda_{;\nu} + \frac{1}{2} \xi^\lambda h_{\mu\nu;\lambda})(x) \bar{\xi}^\omega(y) > \quad (188)
\end{aligned}$$



with the gauge dependent part by (173) and (112):

$$\begin{aligned}
& -\frac{1}{2}\frac{1}{\alpha}\kappa^2 \int d^4x\sqrt{-g} \int d^4y \int d^4w \int d^4z\sqrt{-g}\sqrt{-g}(\mathcal{G}^{\mu\nu} - \mathcal{T}^{\mu\nu})(x) \\
& < (h_{\omega\tau;\tau} - \frac{1}{2}h_{\tau;\omega}^{\tau})(y)(h_{\mu\lambda}\xi^{\lambda}_{;\nu} + h_{\nu\lambda}\xi^{\lambda}_{;\mu} + \xi^{\lambda}h_{\mu\nu;\lambda})(x)\bar{\xi}_{\sigma}(w) > \mathcal{F}^{\sigma\lambda}(w) \\
& \Pi_{\lambda\rho}(w, z)\mathcal{F}^{\sigma\rho}(z) < \xi_{\sigma}(z)\bar{\xi}^{\omega}(y) >
\end{aligned} \tag{189}$$

that cancels with (182). Also (181) contains, cp. (188):

$$\begin{aligned}
& -\frac{1}{\sqrt{\alpha}}\kappa^2 \int d^4x\sqrt{-g} \int d^4y \int d^4w \int d^4u \int d^4t\sqrt{-g}(\mathcal{G}^{\mu\nu} - \mathcal{T}^{\mu\nu})(x) \\
& \mathcal{F}_{\sigma\omega}(y)^4\sqrt{-g}\Pi^{\sigma\tau}(y, w)^4\sqrt{-g} < A_{\tau}(w)F_{\xi\eta}(u) > \\
& (g^{\alpha\xi}\mathcal{F}^{\beta\eta} - \frac{1}{4}g^{\alpha\beta}\mathcal{F}^{\xi\eta})(u)\Pi^{g_{\alpha\beta}(u)}_{g_{\gamma\delta}(t)} < h_{\gamma\delta}(t)(h_{\mu\lambda}\xi^{\lambda}_{;\nu} + \frac{1}{2}\xi^{\lambda}h_{\mu\nu;\lambda})(x)\bar{\xi}^{\omega}(y) >
\end{aligned} \tag{190}$$

with no gauge parameter dependence, keeping in mind that the normalization of the ghost propagator involves the gauge parameter  $\alpha$ .

All dependence on the gauge parameters  $\alpha$  and  $\beta$  cancels out so far. We then turn to terms constructed also from the coupling (118).

With two couplings (118) one gets:

$$\begin{aligned}
& \int d^4w \int d^4u R^{g_{\xi\eta}(w)}_{\xi_{\mu}(x)} G_{\xi\eta, \alpha\beta}(w, u) < h_{\alpha\beta}(u)h_{\gamma\delta}(y) > \\
& \rightarrow i\frac{1}{\sqrt{\alpha}}\kappa^2 \int d^4w\sqrt{-g} < (\omega_1\mathcal{A}_{\mu}A^{\kappa}_{;\kappa} + \omega_2\mathcal{F}_{\kappa\mu}A^{\kappa})(x)(\omega_1\mathcal{A}^{\rho}A^{\epsilon}_{;\epsilon} + \omega_2\mathcal{F}^{\epsilon\rho}A_{\epsilon})(w) > \\
& < (\xi_{\gamma;\delta} + \xi_{\delta;\gamma})(y)\bar{\xi}_{\rho}(w) >
\end{aligned} \tag{191}$$

which does not contribute to (181), while its contribution to (180) is:

$$\begin{aligned}
& \frac{1}{\alpha}\kappa^2 \int d^4x\sqrt{-g}(\mathcal{G}^{\mu\nu} - \mathcal{T}^{\mu\nu})(x) \\
& \int d^4y\sqrt{-g} \int d^4w\sqrt{-g} < (\omega_1\mathcal{A}_{\tau}A^{\kappa}_{;\kappa} + \omega_2\mathcal{F}_{\kappa\tau}A^{\kappa})(y)(\omega_1\mathcal{A}^{\rho}A^{\epsilon}_{;\epsilon} + \omega_2\mathcal{F}^{\epsilon\rho}A_{\epsilon})(w) > \\
& (< \xi_{\lambda;\mu}(x)\bar{\xi}_{\rho}(w) > < \xi^{\lambda}_{;\nu}(x)\bar{\xi}^{\tau}(y) > -R_{\sigma\mu\lambda\nu}(x) < \xi^{\sigma}(x)\bar{\xi}_{\rho}(w) > < \xi^{\lambda}(x)\bar{\xi}^{\tau}(y) >)
\end{aligned} \tag{192}$$

and (192) cancels with (174). Also (192) cancels with (184) for  $\omega_2 = 1$  and a transverse photon propagator (Landau-DeWitt gauge).

By (33) and (183) one finds, forming the two-point correlation function  $< A_{\alpha}(t)h_{\lambda\rho}(z) >$  by means of the coupling (118):

$$\begin{aligned}
& \int d^4y \int d^4z \int d^4w \int d^4u \int d^4t R^{g_{\mu\nu}(x)}_{\xi_{\omega}(y), g_{\lambda\rho}(z)} \\
& (N^{\xi_{\omega}(y)\xi_{\sigma}(w)} R^{A_{\xi}(u)}_{\xi_{\sigma}(w)} G_{\xi, \alpha}(u, t) < A_{\alpha}(t)h_{\lambda\rho}(z) > \\
& + N^{\xi_{\omega}(y)c(w)} R^{A_{\xi}(u)}_{c(w)} G_{\xi, \alpha}(u, t) < A_{\alpha}(t)h_{\lambda\rho}(z) >)
\end{aligned}$$

$$\begin{aligned}
& \rightarrow \frac{\omega_2}{\alpha} \kappa^4 \int d^4 y \sqrt{-g} \int d^4 w \int d^4 u \sqrt{-g} \mathcal{F}_{\omega\sigma}(y)^4 \sqrt{-g} \Pi^{\omega\tau}(y, w)^4 \sqrt{-g} < A_\tau(w) A_\kappa(u) > \mathcal{F}^{\kappa\rho}(u) \\
& (< (\xi_{\mu;\lambda} + \xi_{\lambda;\mu})(x) \bar{\xi}_\rho(u) > < \xi^\lambda{}_{;\nu}(x) \bar{\xi}^\sigma(y) > + < (\xi_{\nu;\lambda} + \xi_{\lambda;\nu})(x) \bar{\xi}_\rho(u) > < \xi^\lambda{}_{;\mu}(x) \bar{\xi}^\sigma(y) > \\
& + < (\xi_{\mu;\nu;\lambda} + \xi_{\nu;\mu;\lambda})(x) \bar{\xi}_\rho(u) > < \xi^\lambda(x) \bar{\xi}^\sigma(y) >) \quad (193)
\end{aligned}$$

contributing to (180) and the trivial part of (181), where only the term  $\delta^{(\mu\nu)}_{(\lambda\rho)}$  of the projection operator  $\Pi^{h_{\lambda\rho}(y)}_{h_{\mu\nu}(x)}$  is kept:

$$\begin{aligned}
& -\frac{2}{\alpha} \omega_2 \kappa^2 \int d^4 x \sqrt{-g} (\mathcal{G}^{\mu\nu} - \mathcal{T}^{\mu\nu})(x) \\
& \int d^4 y \sqrt{-g} \int d^4 w \int d^4 u \sqrt{-g} \mathcal{F}_{\omega\sigma}(y)^4 \sqrt{-g} \Pi^{\omega\tau}(y, w)^4 \sqrt{-g} < A_\tau(w) A_\kappa(u) > \mathcal{F}^{\kappa\rho}(u) \\
& (< \xi_{\lambda;\mu}(x) \bar{\xi}_\rho(u) > < \xi^\lambda{}_{;\nu}(x) \bar{\xi}^\sigma(y) > - R^\sigma{}_{\mu\lambda\nu}(x) < \xi_\sigma(x) \bar{\xi}_\rho(u) > < \xi^\lambda(x) \bar{\xi}^\sigma(y) >). \quad (194)
\end{aligned}$$

Also using both the couplings (74) and (118) to form a graviton two-point correlation function of second order in  $\kappa$  one gets:

$$\begin{aligned}
& \int d^4 w \int d^4 u R^{g_{\xi\eta}(w)}_{\xi_\mu(x)} G_{\xi\eta,\alpha\beta}(w, u) < h_{\alpha\beta}(u) h_{\gamma\delta}(y) > \\
& \rightarrow i \omega_2 \kappa^{24} \sqrt{-g} \int d^4 w \int d^4 z < \xi_\lambda(w) \bar{\xi}_\mu(x) > \mathcal{F}^{\lambda\omega}(w)^4 \sqrt{-g} \Pi_{\omega\delta}(w, z)^4 \sqrt{-g} \mathcal{F}^{\delta\gamma}(z) \\
& < (\xi_{\alpha;\beta} + \xi_{\beta;\alpha})(y) \bar{\xi}_\gamma(z) > \\
& + i \frac{\omega_2}{\sqrt{\alpha}} \kappa^2 \sqrt{-g} \int d^4 w \sqrt{-g} < h_{\alpha\beta}(y) h_{\lambda\rho}(w) > (\mathcal{F}^{\lambda\omega} g^{\rho\sigma} - \frac{1}{4} g^{\lambda\rho} \mathcal{F}^{\sigma\omega})(w) < F_{\sigma\omega}(w) A^v(x) > \mathcal{F}_{v\mu}(x) \quad (195)
\end{aligned}$$

contributing to (180) and the trivial part of (181) first:

$$\begin{aligned}
& \frac{2\omega_2}{\sqrt{\alpha}} \kappa^2 \int d^4 x \sqrt{-g} (\mathcal{G}^{\mu\nu} - \mathcal{T}^{\mu\nu})(x) \\
& \int d^4 w \sqrt{-g} < h_{\tau\rho}(w) (h_{\mu\lambda} \xi^\lambda{}_{;\nu} + \frac{1}{2} \xi^\lambda h_{\mu\nu;\lambda})(x) \bar{\xi}^\kappa(y) > \\
& (\mathcal{F}^{\tau\omega} g^{\rho\sigma} - \frac{1}{4} g^{\tau\rho} \mathcal{F}^{\sigma\omega})(w) < F_{\sigma\omega}(w) A^v(x) > \mathcal{F}_{v\mu}(x) \quad (196)
\end{aligned}$$

canceling with (175). For  $\omega_2 = 1$  and a transverse graviton propagator (Landau-DeWitt gauge) (196) cancels with (188) and (190). One also gets from (195) the following contribution to (180) and the trivial part of (181):

$$\begin{aligned}
& -2\omega_2 \kappa^2 \int d^4 x \sqrt{-g} (\mathcal{G}^{\mu\nu} - \mathcal{T}^{\mu\nu})(x) \\
& \int d^4 w \int d^4 z < \xi_\lambda(w) \bar{\xi}_\mu(x) > \mathcal{F}^{\lambda\sigma}(w)^4 \sqrt{-g} \Pi_{\sigma\omega}(w, z)^4 \sqrt{-g} \mathcal{F}^{\kappa\omega}(z) \\
& (< \xi_{\lambda;\mu}(x) \bar{\xi}_\rho(w) > < \xi^\lambda{}_{;\nu}(x) \bar{\xi}^\tau(y) > - R^\sigma{}_{\mu\lambda\nu}(x) < \xi_\sigma(x) \bar{\xi}_\rho(w) > < \xi^\lambda(x) \bar{\xi}^\tau(y) >). \quad (197)
\end{aligned}$$

(181) also contains contributions from the nontrivial part of the projection operator (69) and from (179) and involving the coupling (118) once. After some calculation one finds that they cancel with (194) and (197).

To summarize, a proof on the formal (non-regularized) level has been sketched in this appendix that the Vilkovisky construction given by (22) makes the effective action of the Maxwell-Einstein system gauge parameter independent at the one-loop level and at next-lowest order in the gravitational coupling constant  $\kappa$ . In the course of the proof it was found that the Landau-DeWitt gauge given by (169), (170) and (171) and with also the  $O(\kappa)$  term included in (171) makes the expressions (180) and (181) vanish, i.e. both the terms of (22) not involving the field space connection, in contrast to the lowest order calculation, where the terms of the gauge condition (171) involving only the field  $h_{\mu\nu}$  are sufficient for this effect.

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